4.2 Notes

From last time we saw that if two functions are linearly independent then one is not a constant multiple of the other

$$y_1 \neq c\, y_2$$

If $y_1 = c y_2 \Rightarrow$ Linearly dependent.

For our discussion let

$$\boxed{\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + q(t)\, y = g(t) \qquad *}$$

$$\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + q(t)\, y = 0 \qquad **$$

$$y'' + p(t)\, y' + q(t)\, y = 0$$

$$y'' + p\, y' + q\, y = 0$$

$y_1(t)$: y_1

$y_2(t)$: y_2

Suppose y_1 & y_2 are solutions to $**$,

Then $y = c_1 y_1 + c_2 y_2$ is also a solution.

$$y' = c_1 y_1' + c_2 y_2'$$

$$y'' = c_1 y_1'' + c_2 y_2''$$ Then

$**$ $\quad y'' + p y' + q y = 0$ and

$$(c_1 y_1'' + c_2 y_2'') + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2) = 0$$

$$(c_1 y_1'' + p c_1 y_1' + q c_1 y_1) + (c_2 y_2'' + p c_2 y_2' + q c_2 y_2) = 0$$

$$c_1 (y_1'' + p y_1' + q y_1) + c_2 (y_2'' + p y_2' + q y_2) = 0$$

but y_1 & y_2 are solns.

$$\therefore \; c_1(0) + c_2(0) = 0.$$

So if y_1 & y_2 are solns. to $**$ then

$y = c_1 y_1 + c_2 y_2$ is also a soln.

for $c_1, c_2 \in \mathbb{R}$.

E & U THM

Given (A) $ay'' + by' + cy = 0$, $y(t_0) = Y_0$
$y'(t_0) = Y_1$

There is a unique solution when

$0 \neq a, b, c, Y_0, Y_1 \in \mathbb{R}$; valid $\forall t \in (-\infty, \infty)$.

THM If $y_1(t)$ & $y_2(t)$ are any two solns to (A) that are linearly independent on $(-\infty, \infty)$ then we can find $c_1, c_2 \in \mathbb{R}$ st $c_1 y_1 + c_2 y_2$ is also a solution to (A).

Lemma If y_1 & y_2 are any two solutions to (A) and $y_1 y_2' - y_1' y_2 = 0$

Then y_1 & y_2 are linearly dependent

Pf by TNM. Given y_1 & y_2 are solns to (A)

let $Y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = Y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = Y_1 \end{cases}$$

Now Play

$$\left.\begin{aligned} &y_2'(t_0)\left(c_1 y_1(t_0) + c_2 y_2(t_0) = Y_0\right) \\ -\,&y_2(t_0)\left(c_1 y_1'(t_0) + c_2 y_2'(t_0) = Y_1\right) \end{aligned}\right\}$$

$$\begin{aligned} c_1 y_1 y_2' + c_2 y_2 y_2' &= y_2' Y_0 \\ - c_1 y_1' y_2 - c_2 y_2 y_2' &= - y_2 Y_1 \end{aligned}$$

$$c_1(y_1 y_2' - y_1' y_2) = y_2' Y_0 - y_2 Y_1 \Rightarrow c_1 = \frac{y_2' Y_0 - y_2 Y_1}{y_1 y_2' - y_1' y_2}$$

by a similar argument we can find c_2

$$c_2 = \frac{Y_1 y_1 - Y_0 y_1'}{y_1 y_2' - y_1' y_2}$$

As a direct consequence of the expression for C_1 & C_2 we have

$$y_1 y_2' - y_1' y_2 \neq 0$$

If $y_1 y_2' - y_1' y_2 = 0$ then by the uniqueness of the derivation it must be that $y_1 = \alpha y_2$ or y_1 & y_2 are linearly dependent.

Also by our E & U THM since we have y_1 & y_2 are solutions to (A)

and $y = c_1 y_1 + c_2 y_2$ is another solution

then <u>every</u> solution can be written

as $y = c_1 y_1 + c_2 y_2$, sometimes

called general soln $(y_h = c_1 y_1 + c_2 y_2)$

Definition: $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

is the Wronskian of y_1 & y_2.

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

if $W[y_1, y_2] \neq 0 \Rightarrow y_1$ & y_2 are linearly indep.

if $W[y_1, y_2] = 0$ then y_1 & y_2 are linearly dependent.

So if we have $y'' + p y' + q y = 0$ and find two solutions y_1 & y_2, if $W[y_1, y_2] \neq 0$ we are done, then general solution is $y = c_1 y_1 + c_2 y_2$.