Infinite Sequences & Series

A sequence is an ordered set of numbers.

$\{a_n\} = a_1 , a_2 , a_3 , \ldots , a_n , \ldots$ where a_n denotes the n^{th} term.

The sequence $\{a_n\}$ converges if $\lim_{n\to\infty} a_n = L$, otherwise the sequence diverges.

Thm 1: If $a_n \le b_n \le c_n \;, n \ge n_0$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$. [Sandwich thm]

Thm 2: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Thm 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, n an integer, then $\lim_{n\to\infty} a_n = L$. [Allows us to use l'Hospital rule]

Thm 4: $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

A sequence is increasing if $a_n \le a_{n+1} \;, n \ge 1$ and decreasing if $a_n \ge a_{n+1} \;, n \ge 1$, and if the sequence is either increasing or decreasing the referred to as monotonic.

A sequence is bdd above it there is a number M st $a_n \le M$, $n \ge 1$ and bounded below if there exists m st $m \le a_n \;, n \ge 1$

If a sequence is bdd above and below, then said to be bdd.

Thm 5: Every bdd, monotonic sequence is convergent.

Series

Given a sequence $\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$, if we add the terms we get a SERIES.

So $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$ denotes an infinite series, where a_n is the n^{th} term.

Spse we have $\sum_{n=1}^{\infty} a_n$, let S_n represent the n^{th} partial sum,

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n,$$

these partial sums form a sequence, $\{S_n\}$.

If $\{S_n\}$ is convergent $\left(\lim_{n\to\infty} S_n = S\right)$ then $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = S$, and S is called the sum of the series, otherwise the series is convergent.

Although it is possible to find the sum of a series, in general this is very difficult and we will be satisfied to determine just convergence or divergence.

** **Geometric Series**** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$

converges if $|r| < 1$ and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$

** **Harmonic Series**** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, DIVERGES

Thm 6: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Beware the converse is false!!!!!!!!!!!!!!

TEST 1: If $\lim_{n\to\infty} a_n \neq 0$ or DNE then $\sum_{n=1}^{\infty} a_n$ is divergent (Divergence Test)

Thm 7: If $\sum a_n$ and $\sum b_n$ are convergent, then so are

(i) $\sum ca_n = c\sum a_n$

(ii) $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$

TESTS FOR CONVERGENCE

Here we are concerned with the convergence or divergence of an infinite series. If we can also determine the sum of the series then fine.

Divergence test: If $\lim_{n\to\infty} a_n \neq 0$ or DNE, then $\sum_{n=1}^{\infty} a_n$ diverges.

Integral test: Spse f is continuous, positive, and decreasing on $[1,\infty)$, let $a_n = f(n)$.

Then $\sum a_n$ is convergent if and only if $\int_1^{\infty} f(x)\, dx$ is convergent.

** **P-series**** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for p > 1 and divergent for $p \leq 1$.

Comparison test: Spse $\sum a_n$, $\sum b_n$ are series with positive terms.

i) If $\sum b_n$ is convergent and $a_n \le b_n, \forall n$ then $\sum a_n$ is also convergent.

ii) If $\sum b_n$ is divergent and $a_n \ge b_n, \forall n$ then $\sum a_n$ is divergent.

Common series used as comparisons are geometric, harmonic, and p-series.

Limit Comparison test: Spse $\sum a_n$, $\sum b_n$ are series with positive terms.

i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then both series converge or diverge.

ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Alternating Series test: If $\sum_{n=1}^{\infty} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$ ($a_n > 0$) satisfies:

i) $a_{n+1} \le a_n \quad , \forall n$

ii) $\lim_{n\to\infty} a_n = 0$

then the series is convergent.

The series $\sum a_n$ is absolutely convergent if the series of absolute values, $\sum |a_n|$ is convergent.

The series $\sum a_n$ is conditionally convergent if it is convergent but not absolutely convergent.

Thm 8: If a series is absolutely convergent, then it is convergent.

Ratio test:

i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent (and thus convergent).

ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or ∞, then $\sum a_n$ is divergent.

Root test:

i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series is absolutely convergent.

ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or ∞, then the series is divergent.

Power Series

A power series is an expression of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

More generally:

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \ldots$$

is a power series centered at "c" or a power series about "c".

One usually determines convergence or divergence of a power series by using the ratio test.

For a given power series, one of following must occur:

i) Convergence at x = c only (R = 0)

ii) Convergence for all x (R = ∞)

iii) There is a positive number R st the series converges if $|x-c| < R$ and diverges for $|x-c| > R$.

R is the <u>radius of convergence</u>. When $x = c \pm R$ (the endpoints) each must be checked individually for convergence (usually by comparison).

Our goal is to find a power series representation for a given function. The trick is to be able to determine the coefficients, a_n.

So spse we have a function f(x), we want:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \quad |x-c| < R$$

then $a_n = \dfrac{f^{(n)}(c)}{n!}$.

Taylor Series:

$$f(x) = \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots$$

When c = 0, we have the special Taylor series called the Maclaurin Series.

Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

Now, we have assumed that a given function has a Taylor series expansion. How do we know? When is it possible for a function to have a Taylor series expansion?

Taylor's Formula: If f(x) has n+1 derivatives in an interval with $c \in I$, then for $x \in I$ there is a number "z", strictly between x and c st

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$.

$R_n(x)$ is called the remainder term.

So $f(x) = T_n(x) + R_n(x)$.

Now f(x) is equal to its Taylor Series expansion on $|x-c| < R$ if $\lim_{n\to\infty} R_n(x) = 0$..