Example(5.1-5.3): Determine the area of the region bounded by $f(x) = 5 - x^2$, x-axis and $-1 \le x \le 2$.

We will need the following: (A) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, (B) $\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$.

$\Delta x = \frac{b-a}{n} = \frac{2-{}^{-}1}{n} = \frac{3}{n}$.

Now $x_0 = -1$, $x_1 = -1 + \Delta x$, $x_2 = -1 + 2\Delta x$, … , $\mathbf{x_i = -1 + i\Delta x}$, … , $x_n = -1 + n\Delta x = 2$.

Then $\Delta x = \frac{3i}{n}$ and the area of the i^{th} rectangle is $A_i = f(x_i)\Delta x$. So

$f(x_i) = f\left(\frac{3i}{n} - 1\right) = \left[5 - \left(\frac{3i}{n} - 1\right)^2\right] = 4 - \frac{9i^2}{n^2} + \frac{6i}{n}$ and $A_i = \frac{12}{n} - \frac{27i^2}{n^3} + \frac{18i}{n^2}$.

Then $A \approx \sum_{i=1}^{n} A_i$ and finally putting in infinitely many subintervals

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} A_i = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{12}{n} - \frac{27i^2}{n^3} + \frac{18i}{n^2}\right)$$

$$= \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \frac{12}{n} - \sum_{i=1}^{n} \frac{27i^2}{n^3} + \sum_{i=1}^{n} \frac{18i}{n^2} \right\} = \lim_{n\to\infty} \left\{ \frac{12}{n} \sum_{i=1}^{n} 1 - \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \frac{18}{n^2} \sum_{i=1}^{n} i \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{12}{n} n - \frac{27}{n^3} \frac{n(2n+1)(n+1)}{6} + \frac{18}{n^2} \frac{n(n+1)}{n} \right\}$$

$$= \lim_{n\to\infty} \left\{ 12 - \frac{9}{2}\left(\frac{2n^3 + 3n^2 + n}{n^3}\right) + 9\left(\frac{n^2 + n}{n^2}\right) \right\}$$

$$= \lim_{n\to\infty} \left\{ 12 - \frac{9}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 9\left(1 + \frac{1}{n}\right) \right\}$$

$$= \{12 - 9 + 9\} = 12$$