MATH 606 - Homework #5– due dates mixed, see below

Problems on this page due: Monday May 23 in class

(1) Show that the Lax-Wendroff method is essentially the only explicit finite-difference method for $u_t = \alpha u_x$ that has the following two properties:

(i) $u_{i,j+1}$ depends only on $u_{i-1,j}$, $u_{i,j}$ and $u_{i+1,j}$

(ii) the global accuracy (the order of the discretization error) is $O(h^2 + k^2)$

You can show this by completing the following steps:

(a) Let

$$u_{i,j+1} = A_1 u_{i+1,j} + A_0 u_{i,j} + A_{-1} u_{i-1,j}$$

and let U be the exact solution of the PDE and u be the solution of the FDE so that

$$U_{i,1} - u_{i,1} = kO(h^2 + k^2)$$

(So here I am asking that you assume $U_{i,0} = u_{i,0}$ for all i.)

(b) Expand U in a Taylor series about (i, j). Substitute this and the expression for u in part (a) into the expression for the discretization error for part (b).

(c) Solve for A_1, A_0, and A_{-1}.

(2) Calculate the dissipation and dispersion relations for:

(i) leapfrog

(ii) Lax-Wendroff

(iii) upwinding

Problems on this page are due by class time, Friday May 27

(3) Solve the hyperbolic PDE $u_t + \alpha u_x = 0$ for $x \in [-3, 3]$ for $\alpha = 1, -1$ with periodic boundary conditions using two different initial conditions:

(i) $u(x, 0) = 1 - |x|$ for $x \in [-1, 1]$, and $u(x, 0) = 0$ otherwise

(ii) $u(x, 0) = 2|x|^3 - 3x^2$ for $x \in [-1, 1]$, and $u(x, 0)$ otherwise

Do this using the three distinct methods

(a) Leapfrog (CTCS)

(b) Upwinding (FTBS or FTFS depending on if α is positive or negative). (note: write your code so that it easily switches automatically between the two depending on the sign of α)

(c) Lax-Wendroff

In all cases, computationally show that the method has the correct order of accuracy for fixed $\rho = 1/2$, varying h (you know the exact solution!). Plot the solutions at t=0,1,2,3,4,5.

Please write your code for a general α and show plots for $\alpha = 1$ and $\alpha = -1$, but you need only study the order of accuracy for $\alpha = 1$ for (a) and (c).

(4) Solve the hyperbolic PDE $u_t + u_x = 0$ for $x \in [0, \infty)$ and $t \geq 0$ with the following initial/boundary values: $u(x, 0) = 1 - |x - 1|$ for $x \in [0, 2]$ and $u(x, 0) = 0$ otherwise, and $u(0, t) = 0$ for $t \geq 0$ using

(a) Wendroff's implicit method

(b) Lax-Wendroff

Study the error for $\rho = .5, 1$, and indicate which combination(s) of ρ and difference scheme gives the best results with a brief explanation as to why.

Please zip all m-files into one file called "hw6" and email to me at heather@math.ohio-state.edu before class time Friday, May 27. Handwritten components and printed results for these problems should be turned in by the beginning of class the same day.