MATH 606 - Homework #3– due May 4, 2005

(1) The equation $u_t = au_{xx}$ for $0 \leq x \leq 1$ and $t \geq 0$ and a a positive constant can be approximated at the (i,j) grid point by the fully implicit backward-difference scheme

$$u_{i,j+1} = ra(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) + u_{i,j}$$

where $r = k/h^2$. Assuming the boundary and initial values are known (Dirichlet boundary values!), show (by hand!) that

(a) The scheme is unconditionally stable using the matrix method

(b) The order of the discretization error is second order accurate in h, but first order accurate in k.

(c) Write a code for this scheme and computationally verify the order of accuracy found in (b). Use zero boundary values and the initial value $u(x, 0) = sin(\pi x)$.

(2) (a) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Calculate $\|A\|_1$, $\|A\|_2$ and $\|A\|_\infty$ by hand.

(b) Consider the matrix

$$B = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

Calculate $\|B\|_1$ and $\|B\|_\infty$ by hand, and $\|B\|_2$ by using the eig function in MATLAB.

(3) Let $f(x) = x^2$ and $g(x) = x^2 + \epsilon sin(kx)$ for $x \in [0 : h : \pi]$ and where k is a positive integer and ϵ is some small number. Show that f and g are "close" by showing that $\|f - g\|$ is much smaller than $\|f\|$ or $\|g\|$. Do this in the 1-norm, 2-norm, and ∞-norm.

(4) Show that the DuFort-Frankel scheme is stable for all r by using Fourier methods. Determine the stability of $u_t = \sqrt{-1}u_{xx}$ with the discretization being leapfrog in time and second-order central in space.

(5) Solve the convection-diffusion equation $u_t + \alpha u_x = \nu u_{xx}$ for $x \in [-1, 1]$ and for all $t \geq 0$ with Neumann boundary conditions $u_x(-1, t) = u_x(1, t) = 0$.

(a) Give and use an implicit method which is second-order accurate in both space and time. Let $\alpha = 1$ for simplicity and use two different values for ν, $\nu = 1.0, 0.01$. Check your code to determine that it is working correctly (be sure you explain to me in writing how you did this!). Be sure to check that your computed solution is close to a true solution, and that your error decreases at the correct rate.

(b) Now you've got your code working, let your initial condition be $u(x, 0) = x^3 - 3x$. Plot the results at various times. Take h small enough so that your plots look smooth.

(c) Now let your initial condition be $u(x, 0) = -(x^3 - 3x)$. Do same as in (b).

Please zip all m-files into one file called "hw3" and email to me at heather@math.ohio-state.edu by 11:00 am Wednesday, May 4. Handwritten files should be turned in by the beginning of class the same day.