MATH 606 - Homework #1 – due April 8, 2005

(1) Given an nx1 array, x, write a Matlab function that accepts x as input and, without using for loops, returns as output a vector y defined in each of the following cases: (Note: you are not allowed to pass n as an input.)

(a) $y_i = x_{n-i+1} \quad \forall i = 1, ..., n$

(b) $y_i = x_{n/2+i} \quad \forall i = 1, ...n/2$ and $y_i = x_{i-n/2} \quad \forall i = n/2+1, ..., n$ (for this part, first check that n is even and return an error if it isn't)

(c) $y_i = x_{i+1} - 2x_i + x_{i-1} \quad \forall i = 1, ..., n$, where periodic boundary conditions are assumed - i.e. $x_0 = x_n$ and $x_{n+1} = x_1$.

(2) Using as few instructions as possible and no for loops, write a Matlab function which takes n as input and generates the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ n+1 & n+2 & n+3 & \dots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n(n-1)+1 & n(n-1)+2 & n(n-1)+3 & \dots & n^2 \end{bmatrix}$$

(3) Consider the integral

$$E_n := \int_0^1 x^n e^{x-1}\, dx$$

for $n = 0, 1, 2...$

(a) Show explicitly using integration by parts that we can replace this definition by the following recursive definition of E_n

$$E_n = 1 - nE_{n-1} \qquad \text{with} \qquad E_0 = 1 - 1/e$$

(b) Write a short computer program (showing results in double precision) to calculate $E_0, E_1, E_2, ..., E_{50}$ using the recursive formula in (a). Your result should not seem "reasonable". Why?

(c) To understand what's happening, carry out the following analytical calculation. Let $\tilde{E}_0 = E_0 + \epsilon_0$ where $\epsilon_0 = \epsilon << 1$. Then $\tilde{E}_1 = E_1 + \epsilon_1$, and $\tilde{E}_2 = E_2 + \epsilon_2$, ... for some ϵ_n's > 0. Calculate ϵ_1, ϵ_2, ..., ϵ_4. What do you conjecture is the relationship between ϵ_n and ϵ_0? Does this explain why your numerical calculation was not "reasonable"?

(4) Asymptotics:

(a) Show that $sin(x) = O(1)$, but that $sin(x) \nsim 1$ as $x \to \infty$

(b) Show that $ln(x) = o(x^m)$ for all integers $m > 0$ as $x \to \infty$

(c) Show that $sinh(x) \sim cosh(x) \sim e^x/2$ as $x \to \infty$

(d) Find the asymptotic behavior of $e^{1/x}$ as $x \to 0$

(5) Let $u(x) = xsin(x)$. Generate the table:

h	$D_h^{(f)}u(1)$	$error^{(f)}$	$D_h^{(c)}u(1)$	$error^{(c)}$	$D_h^{(2f)}u(1)$	$error^{(2f)}$
$.1$						
$.1/2$						
$.1/2^2$						
$.1/2^3$						
$.1/2^4$						
$.1/2^5$						
$.1/2^6$						
$.1/2^8$						
$.1/2^{10}$						

where:

$$D_h^{(f)}u(x) := \frac{u(x+h) - u(x)}{h}$$

$$D_h^{(c)}u(x) := \frac{u(x+h) - u(x-h)}{2h}$$

$$D_h^{(2f)}u(x) := \frac{-u(x+2h) + 4u(x+h) - 3u(x)}{2h}$$

and $error(\cdot) = u'(1) - D_h^{(\cdot)}u(1)$. Please print the value of the D's to at least 10 decimal digits and the errors in scientific notation.

(6) Consider the "standard" second-order central difference approximations for $f'(x)$ and $f''(x)$, namely

$$f'(x) \sim \frac{f(x+h) - f(x-h)}{2h} \text{ and } f''(x) \sim \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

(Note: know how to derive these!) For fixed x, show that you obtain the same approximation if you take $p(x)$ to be the unique quadratic polynomial which passes through the three points $(x-h, f(x-h))$, $(x+h, f(x+h))$, $(x, f(x))$ and instead calculate $p'(x)$ and $p''(x)$.