

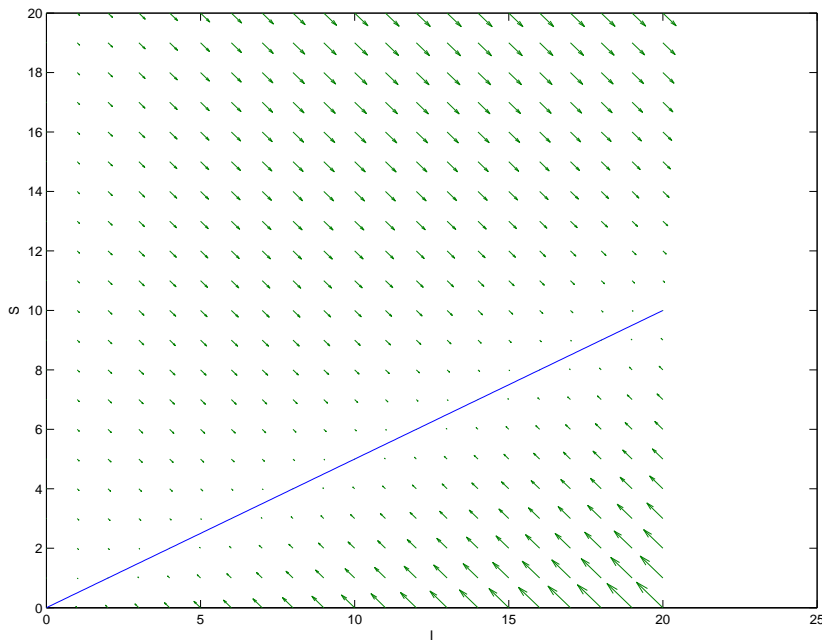
Solution:

We are analyzing the system

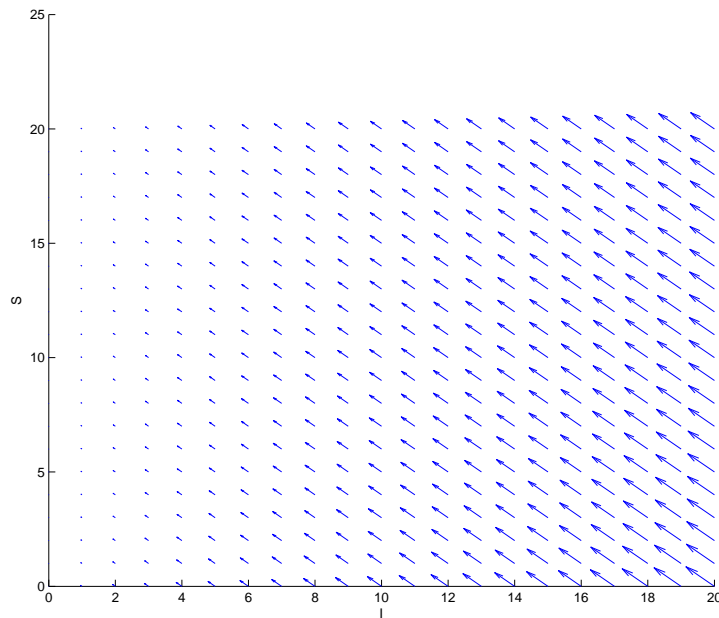
$$\begin{aligned}\frac{dS}{dt} &= \frac{-bSI}{S+I} + gI \\ \frac{dI}{dt} &= \frac{bSI}{S+I} - gI\end{aligned}$$

The nullclines are found by setting $\frac{dS}{dt} = 0$ and $\frac{dI}{dt} = 0$. We get from both equations that either $I = 0$ or $\frac{-bS}{S+I} + g = 0$. The second can be manipulated to look like $S = \frac{-g}{g-b}I = \frac{g}{b-g}I$. Since we get the same nullclines from both equations, when we take their intersections, we get the entire nullclines as our equilibria. So any pair of populations of the form $(I, S) = (0, k)$ or $(I, S) = (k, \frac{g}{b-g}k)$ are equilibria for all nonnegative real numbers k . The phase plane looks different depending on whether $b > g$ or $b < g$. Notice that if $b = g$, we have only the nullcline $I = 0$.

Phase plane for $b > g$ (here took $b = 3$ and $g = 1$)



Phase plane for $b < g$ (took $b = 1$ and $g = 3$)



(notice the only nullcline for this case that falls in the first quadrant is $I=0$)

The b represents the rate at which susceptibles are recruited into the infected class - so could measure a likelihood of infection upon contact with an infected. The g represents the rate of recovery of an infected individual. Notice that if you add $\frac{dS}{dt}$ and $\frac{dI}{dt}$ you get

$$\frac{d(S + I)}{dt} = 0$$

which tells us that in this model the total population $S + I$ is assumed to be constant. You can also tell this is the case because if infecteds are lost due to recovery ($-gI$ term in dI/dt) they all show up as newly susceptible ($+gI$ term in dS/dt). A similar argument shows that any susceptibles who become infected are all moved into the infected class (rather than dying, for example). There is no birth or death accounted for here.

Notice by the phase plane analysis that if $b < g$, you always end up with no infecteds and all susceptibles at equilibrium (notice this is reasonable since $b < g$ means roughly that the rate at which the disease is spread is slower than the recovery rate). If $b > g$, you will end up with some coexistence steady state where the number of infected individuals is I and the number of susceptibles is $S = \frac{g}{b-g}I$. So, if we are hoping to eliminate the disease from the population, we want $b < g$. Thus, this models tells us that we should probably focus our attention on reducing the likelihood of disease transmission if we want to eradicate the disease (thus reducing b) or on increasing the rate of recovery (so increasing g) whichever is more feasible.. or both!