# CHAPTER 1. INTRODUCTION

This course is an introduction to abstract mathematics. You will learn some basic terminology (the language of sets, relations, and functions), some fundamental proof techniques (deduction and induction), and some important concepts common to all higher mathematics courses (fields, orderings, and completeness). You will, however, need to learn much more than just concepts and techniques. You must also learn how to use these ideas to solve problems. We shall accomplish this goal by asking you to construct proofs.

You may be asking: Why do mathematicians, computer scientists, and statisticians need to learn to prove things? If someone wants to work in industry, why does one need to learn how to construct a proof? The answer to that question could fill a book. I will show restraint by mentioning just two reasons.

First reason: People who use mathematics in their work often encounter situations for which the theory has not yet been worked out. For example, you might design a program which is quite different from those used by others. You want to be sure that your program works (i.e., gets the right answer) and is better than anything else available (for example, it might use less storage or it might require fewer computations). In order to do that, you will have to analyze the program, and this activity will use the very same techniques learned constructing proofs.

Second reason: People who use mathematics sometimes come up with an idea so useful that they want to patent it. In order to get a patent, you need to prove that your idea is correct. You may claim that if the idea solves some problems then it has to be correct. This pragmatic approach is not sufficient to obtain a patent. Why? Because those who set up the U.S. Patent Office know what every working mathematician knows. We are easily fooled! The only way we can be sure that we have not been fooled, claiming to know more than we do, is by subjecting our claims to cold, hard, unforgiving logical argument. Proof plays the same role for users of mathematics that experimentation plays for scientists. Without such corroborating evidence, one claim is as good as another.

Just as no house can be built without a foundation, no rigorous study of mathematics can be made without undefined concepts (primitive notions on which the theory is based) and postulates or axioms (properties which are assumed to be true and need no proof). We shall begin with a rather loose approach, using real numbers and sets as primitive concepts and specifying no axioms. In particular, for the first three chapters you may use any property about real numbers you know, e.g., the commutative laws, the distributive law, even things like the square of a negative number is a positive number. After getting used to constructing proofs in this free and somewhat unstructured environment, we shall begin to introduce axioms for the real number system and proceed with greater caution. We shall continue, however, to treat sets as a primitive concept. (The axioms of set theory are beyond the scope of an introductory class, being both subtler and less easily motivated than those for the real number system.)

For the purist who dislikes the approach outlined above, let me remind you that this is the way mathematics developed. Mathematicians proved theorems for hundreds of years before they even attempted an axiomatic development of the real number system. Moreover, even after the real number system was completely understood it was several decades before anyone even noticed that set theory needed an axiomatic description.

## NUMBERS AND PROOF.

**1.1 Real numbers and integers.** We shall denote the collection of *real numbers* by $\mathbf{R}$ or by $(-\infty, \infty)$. Each real number has a decimal expansion. $\mathbf{R}$ satisfies *closure properties*: The sum, difference, and product of any pair of real numbers is again a real number. The quotient of two real numbers is also a real number provided the divisor is not zero. Division by zero is not allowed.

The collection of real numbers contains several important "number systems."

1. The *natural numbers* or *positive integers*, $1, 2, 3, \dots$, which we denote by $\mathbf{N}$.
2. The integers, $\dots, -2, -1, 0, 1, 2, \dots$, which we denote by $\mathbf{Z}$.
3. The *rationals*, numbers of the form $m/n$ where $m$ and $n$ are integers but $n \neq 0$. We shall denote the set of rational numbers by $\mathbf{Q}$.

We call these number systems because they satisfy their own set of closure properties: The sum and product of natural numbers is a natural number; the sum, difference, and product of two integers (respectively, rationals) is an integer (respectively, a rational); and the quotient of two rationals is a rational, provided the divisor is not zero. We note that positive rational numbers are also called *fractions*.

There is another special type of real number which played a crucial role in the historical development of our concept of number: A real number is said to be *irrational* if it is not a rational. Recall that irrational numbers are characterized by the fact that their decimal expansions neither repeat nor terminate. Thus the number $0.1212\dots$ is rational, but the number $0.112123123412345\dots$ is irrational. Other irrationals include $\sqrt{2}$, $\sqrt{3}$, $\sqrt{3} - \sqrt{2}$, $\pi$, and the natural base $e$.

**1.2 Hypotheses and conclusions.** We shall be dealing *statements* or *propositions*, i.e., with sentences which are either true or false but not both. Many of the statements which occur in mathematics contain *if...then* clauses, i.e., contain statements of the form "if $X$, then $Y$," where $X$ and $Y$ are sentences or clauses. In such a statement, $X$ is called the *hypothesis* and $Y$ is called the *conclusion*. For example, consider the following statements:

$A$) if $x$ is a boy, then $x$ is human

$B$) if $x$ is a tiger, then $x$ is mammal.

The hypothesis of statement $A$ (respectively, $B$) is "$x$ is a boy" (respectively, "$x$ is a tiger") and the conclusion of statement $A$ (respectively, $B$) is "$x$ is human," respectively, "$x$ is a mammal").

Any statement can be written in several different, but equivalent, ways. For example, statement $A$ above can be written as: 1) All boys are human; 2) $x$ is a boy implies that $x$ is human; 3) $x$ is human if $x$ is a boy. In particular, the first clause in a statement is not always its hypothesis, and hypotheses are not always preceded by "if."

**1.3 Counterexamples.** Propositions can be true or false (or neither, but we shall hide that dirty secret from you for several more semesters). Propositions $A$ and $B$ above are both true, but

$C$) if $x$ is a student at UTK, then $x$ lives in Knoxville

is false. There must be several students at UTK who live in Oak Ridge, for example.

When we want to "prove" that a proposition is false, we do not need to construct a proof. All we need to do is produce a *counterexample*, i.e., a specific example which shows the implication to be invalid. For example, to show that proposition $C$ above is false, we need to identify a SPECIFIC student who goes to UTK but does not live in Knoxville. The vague statement we made beginning with "There must be several students..." is not sufficient to show that $C$ is false.

Don't get worried. Since we shall be dealing with propositions involving real numbers, showing that a proposition is false will not mean that you will be asking people on the street where they live. Your counterexamples will come from your knowledge of real numbers.

For example, is the proposition

$D$) if $x$ is negative, then $x^4$ is negative

true or is it false? You probably know from previous courses that the square of a nonzero real number is positive and that the fourth power of a number is the square of its square. This is true, but much too complicated to answer our question. If you believe a proposition to be false, produce a CONCRETE counterexample. In this case, if we let $x = -1$, then $x^4 = 1$ is NOT negative. This, then, "proves" that proposition $D$ is false.

On the other hand, if we want to prove that a mathematical proposition is true, we cannot just look at examples. We must construct a general proof which uses definitions and propositions we have previously proved. (There is one situation when we can use examples to prove a proposition: when the hypotheses are so restrictive that they only hold for finitely many objects. In this case, we can verify that the proposition is true by laboriously checking that each object which satisfies the hypothesis does indeed satisfy the conclusion. Such a "proof" might be best left to a computer.)

**1.4 Negations.** The *negation* of a proposition $X$ is a proposition which has the opposite meaning of $X$. We shall denote the negation of a statement $X$ by *not* $X$.

The negation of most propositions can be accomplished by inserting a "not" at an appropriate point. For example, the negation of "$x$ is a boy" is "$x$ is not a boy." Don't make any unnecessary assumptions when you write down the negation of a statement. Indeed, "$x$ is not a boy" is NOT equivalent to "$x$ is a girl" unless it is understood that $x$ is appriori a human. For example, if $x$ is a rock then $x$ is most certainly neither a girl nor a boy.

In most cases, we will insist that you write your negations as explicitly as possible because doing so conveys more information and is more useful for constructing proofs. For example, if $x$ is a real number, then the negation of "$x < 2$" is "$x$ is not less than 2," but we will insist that you write the negation as "$x \geq 2$." As we shall see below, this approach will be absolutely essential when the proposition contains any of the words or phrases *and, or, for all,* and *there exists.* In particular, writing the negation of "for all real numbers $x$ there exists a real number $y$ such that $x \leq y$" as "for all real numbers $x$ there does NOT exist a $y$ such that $x \leq y$" will be **unacceptable**. For a correct way to negate statements like this, see Sections 1.13 and 1.40 below.

**1.5 Converses and contrapositives.** The *converse* of a proposition "if $X$, then $Y$" is the proposition "if $Y$, then $X$". The *contrapositive* of "if $X$, then $Y$" is the proposition "if not $Y$, then not $X$". These concepts look similar, but logically speaking, there is a great difference between a converse and a contrapositive.

If a proposition "if $X$, then $Y$" is true, then its contrapositive is also ALWAYS true. For example, think about proposition $A$ above. Its contrapositive is "if $x$ is not human, then $x$ is not a boy." Obviously true. If ALL boys are human, then no "NOT human" can be a boy.

In sharp contrast, the converse of a true proposition might be true, but it also might very well be false. Consider proposition $A$ again. Its converse "if $x$ is human, then $x$ is a boy" is obviously false. (To verify that this is the case, we need to produce a concrete counterexample. Here it is: My mother was human, but she was not a boy.)

To further drive home the point that the converse of a proposition is logically independent from the original proposition, notice that the converse of a false proposition might be true or it might be false. For example, proposition $C$ above is false, and so is its converse: If $x$ lives in Knoxville, then $x$ is a student at UTK. (I am a counterexample to that proposition!) On the other hand "if $x$ is human, then $x$ is a girl" is false, but its converse is true.

The fact that the contrapositive of a proposition and the original proposition are logically equivalent can be used to your advantage: Sometimes it is easier to prove the contrapositive of a proposition than the proposition itself. (This is especially true when the proposition involves not's and logical negatives.)

**1.6 Example.** *If $x$ is an irrational of the form $x = n + 1$, prove that $n$ cannot be an integer.*

PROOF. We prove the contrapositive. If $n$ IS an integer, then $x = n + 1$ is also an integer, hence NOT irrational. ∎

**1.7 If and only if propositions.** When a proposition AND its converse are both true, we call such a proposition a *tautology* and abbreviate it by using the phrase *if and only if.* For example, one of the most useful tools for finding roots of polynomials is the tautology

$$ab = 0 \text{ if and only if } a = 0 \text{ or } b = 0.$$

What this means is that if $ab = 0$, then at least one of the numbers $a$ or $b$ is zero, and conversely, if either one (or both) of the numbers $a$ or $b$ is zero, then the product $ab$ is also zero.

Notice, then, that any proof of an "if and only if" proposition has two halves. One must prove that the hypothesis implies the conclusion, and that the conclusion implies the hypothesis.

**1.8 Quantifiers.** Many mathematical propositions involve information which specifies how many things satisfy the given implication. Such information is often given using one of the following two *quantifiers*:

1. *there is* (sometimes written *there are* or *there exist(s)*)—meaning AT LEAST ONE,
2. *for all* (sometimes written *for every* or *given*)—meaning every single one with NO EXCEPTIONS.

It is important for you to recognize from the very beginning that proofs verifying statements which involve "there exists" are very different from those which involve "for all." To prove that THERE EXISTS a $y$ with a property $\mathcal{P}$ we must demonstrate how to find such a $y$. (Sometimes this is accomplished by specifying $y$ using a formula—see Example 1.9 below. At other times this is accomplished by appealing to some fundamental axiom.)

To prove that a property $\mathcal{P}$ holds FOR ALL $x$ we must show that $\mathcal{P}$ holds for a general $x$, that is for every single possible $x$. In particular, if any restrictions are made on $x$ besides the restrictions imposed by the hypotheses, the proof of "for all" is invalid. Let's demonstrate the difference by an example.

**1.9 Example.** *Which of the following propositions is true?*

*E*) For all real numbers $x$ there is an integer $y$ such that $y < x$

*F*) There is an integer $y$ such that $y < x$ for all real numbers $x$

PROOF OF $E$. Let $z = [x]$ be the greatest integer in $x$. (For example, if $x = \pi$, then $z = 3$. If $x = -3.71816$, then $z = -4$.) By definition, $z$ is an integer and

$z \le x < z + 1$. Set $y = z - 1$. Then $y$ is an integer and $y = z - 1 < z \le x$. In particular, $y < x$.

FALSE PROOF OF STATEMENT $F$. Let $x = \pi$ and set $y = 3$. Then $y < x$. What's wrong with this proof? We specified $x$ by letting $x = \pi$. But statement $F$ claims that $y < x$ holds for ALL $x$, not just $\pi$.

CORRECT SOLUTION TO STATEMENT $F$. This statement is false. If such a special $y$ exists, then any real number including $x = y \pm 1$ would have to satisfy $y < x$. For $x = y + 1$ this is correct, but for $x = y - 1$, it is NOT correct: $x = y - 1 < y$. Therefore, statement $F$ is false. ∎

**1.10 WARNING.** *Quantifiers in mathematical propositions are highly noncommutative.*

For example, statements $E$ and $F$ above are identical with one exception: the "for all" in statement $E$ precedes "there exists," whereas in statement $F$ "there exists" precedes "for all." Yet, $E$ is true but $F$ is false!

**1.11 Example.** *Prove that the following proposition is true.*

$G$) Given an irrational $x$, there is an irrational $y$ such that $y - x$ is an integer.

In view of Warning 1.10, we know better than to change the order of these quantifiers in statement $G$. Nevertheless, there are many other ways to misinterpret this proposition. Here are two more:

1) $y - x$ is an integer for all irrationals $x$ and $y$. This cannot be true since $\sqrt{2}$ and $\sqrt{3}$ are irrational but $\sqrt{3} - \sqrt{2}$ is not an integer.
2) $y - x$ is an integer for some irrationals $x$ and $y$. This proposition IS true, for example $(\sqrt{2} + 1) - \sqrt{2} = 1$, but it is NOT equivalent to the original proposition.

The phrase "given any irrational $x$" means "for all irrationals $x$." Thus the correct interpretation of proposition $G$ is as follows. *For all irrational numbers $x$, there is an irrational number $y$* (which in general depends on $x$) *such that $y - x$ is a positive integer.* A correct proof would begin with the statement "Let $x$ be an irrational number." Then we would have to show how to manufacture an irrational $y$ using $x$ (without making any additional assumptions about $x$) such that $y - x$ is an integer. One choice for $y$ would be $y = x$, for then $y - x = 0$ is always an integer. ∎

**1.12 More examples.** Evidently, you must be able to understand the secret language of quantifiers in order to interpret propositions and to construct correct proofs. To further explore the difference between and the meaning of the two standard quantifiers, consider the following two statements:

$H$) There is a quadratic function $f$ such that $f(2) < f(3)$.

$I$) Given a quadratic function $f$, $f(2) < f(3)$.

Statement $H$ is true because it only asks whether one such function exists. (Yes it does, e.g., $f(x) = x^2$.) Statement $I$ is false because the word "given" means that EVERY quadratic satisfies $f(2) < f(3)$. This is evidently false (e.g., $f(x) = 5 - x^2$ is a quadratic but $f(2) = 1$ is NOT less than $f(3) = -4$).

To make matters worse, sometimes the quantifier "for all" can be implicit when variables are used. For example, consider the following proposition: *Suppose that* $f(x) = 3x + 5$. *If* $f(a) < 0$ *and* $f(b) > 0$, *then there is a* $c$ *between* $a$ *and* $b$ *such that* $f(c) = 0$. Although no quantifier is explicitly associated with $a$ and $b$, it is understood by the notation that what is meant is that FOR EVERY pair of real numbers $a$ and $b$ which satisfy $f(a) < 0 < f(b)$, there is a $c$ between $a$ and $b$ such that $f(c) = 0$. Hence a proof which begins with "let $a = -2$ and $b = 2$" will not prove the proposition but only a special case.

**1.13 Negating quantifiers.** In order to negate a statement that contains quantifiers, one adds a "not" at an appropriate point, but also replaces the quantifier "there is" by "for all," and vice versa. For example, the negation of the proposition "there is a real number $x$ such that $x < -n$ for all $n \in \mathbf{N}$" is "for EVERY real number $x$ there is an $n \in \mathbf{N}$ such that $x \geq -n$" (think about it).

Again, we insist that you write out negations explicitly. Negating *There is a real number* $y$ *such that* $x > y$ *for all real numbers* $x$ by writing "there is no $y$ such that $x > y$ for all real numbers $x$" is **unacceptable**. The correct negation of this statement is: "For all real numbers $y$, there is a real number $x$ such that $x \leq y$."

**1.14 Proof types.** There are several types of proof which you may have seen in a previous course, e.g., high school geometry.

A *deductive* proof of a statement "$X$ implies $Y$" begins by assuming $X$, then proceeds step by step until $Y$ is verified. Each step must be justified by a hypothesis, a definition, or a proposition already proved. This is the way we proved statement $E$ above.

An *inductive proof* is a special technique for proving statements involving integers. We will discuss this type of proof thoroughly in Chapter 4.

A *proof by contradiction* begins by assuming $X$ and not $Y$, then proceeds step by step as in a deductive proof, but the aim is to obtain a *contradiction*, i.e., a conclusion which is either obviously false or contrary to the assumptions made. At this point, the proof is complete. The phrase "suppose not" or "suppose to the contrary" always indicates a proof by contradiction.

Proofs by contradiction are also called *indirect proofs*. Most theorems have many different proofs, some direct and some indirect. If you have a choice, the direct proof (which gives more information) is preferable. In some cases you have no choice, or the direct proof is so much more complicated that it is prudent to give the indirect proof anyway.

The following is an illustration of an indirect proof involving quantifiers.

**1.15 Example.** *Given an irrational* $x$, *prove that* $x + x$ *is also irrational.*

SOLUTION. Here is a false proof given by many beginning students. Suppose not, i.e., the sum is rational. Let $x = \sqrt{2}$. Then $x + x = 2\sqrt{2}$ which IS irrational, a contradiction. Hence the theorem is true.

What is wrong with this "proof"? The quantifier was not negated correctly. The proposition says "FOR ALL irrationals $x$, $x+x$ is irrational." Hence the negation of the proposition is "there IS (at least, but possibly ONLY, one) irrational $x$ such that $x+x$ is rational". Since there is no guarantee that the special irrational is $x=\sqrt{2}$ the false proof lost generality at that point.

Here is a correct proof. Suppose not, i.e., there is some irrational $x$ such that $x+x$ is rational. Thus $2x = x+x = p/q$ for some integers $p,q$ with $q \neq 0$. Solving this equation for $x$, we obtain $x = p/(2q)$. Thus $x$ is also rational, a contradiction. ∎

## EXERCISES.

1. Let $x$ be a real number and $p$ be a rational. If $x+p$ is irrational, prove that $x$ is irrational. Does this implication still hold if we replace $x+p$ with $xp$?
2. Negate the following propositions.
   a) For all $\epsilon > 0$ there is an integer $n$ such that $1/n < \epsilon$.
   b) For all real numbers $x$ there is an integer $n$ such that $x < n$.
   c) For all irrationals $x$ and $y$, the sum $x+y$ is irrational.
3. For each of the following propositions, identify its hypotheses and conclusion, and write out its converse and contrapositive.
   a) Given a real number $x$, there is a positive integer $n$ such that $n > x$.
   b) A real number $x$ is irrational if $x+5$ is irrational.
   c) A real number $x$ satisfies $x \geq 1$ when $x$ is a natural number.
   d) If $n \in \mathbf{Z}$, then $n \leq -1$ is always true provided $n < 0$ also holds.
4. Prove that given $\epsilon < 0$, there is a real number $a$ such that $a > \epsilon$.
5. Prove that for all $\epsilon > 0$ there is an $a > 0$ such that $a < \epsilon$.
6. Suppose that $f(x) = x^3 + 2$. Prove that if $f(a) < 0$ and $f(b) > 0$, then there is a $c$ between $a$ and $b$ such that $f(c) = 0$.
7. Decide which of the following propositions are true and which are false. Give reasons (but not formal proofs) for your decisions.
   a) There is a positive integer $n$ such that $n \leq m$ for all positive integers $m$.
   b) There is a real number $x$ such that $x < y$ for all real numbers $y$.
   c) For all integers $x$ there is a positive integer $y$ such that $x < y$.
   d) For all positive integers $y$ there is an integer $x$ such that $x < y$.
   e) Given a real number $y$ there is an integer $x$ such that $x < y$.
   f) Given a real number $y$ there is an integer $x$ such that $x < y + \epsilon$ for all $\epsilon > 0$.

## BASIC DEFINITIONS.

**1.16 Set notation.** If $A$ is a set,

$$x \in A$$

means $x$ is an *element* of $A$ or $x$ *belongs to* $A$. The negation of $x \in A$ will be denoted by

$$x \notin A.$$

Thus $x \notin A$ means $x$ is not an element of $A$ or $x$ does not belong to $A$. For example, $1 \in \mathbf{N}$, $-3 \in \mathbf{Z}$, and $\pi \notin \mathbf{Z}$.

The set with no elements, called the *empty set*, will be denoted by the Danish letter $\emptyset$. Thus the statement $x \in \emptyset$ is ALWAYS false.

Specific sets are often denoted using curly brackets which enclose an explicit description of that set. For example, the set of real numbers $x$ which satisfy $x^2 = 1$ is denoted by

$$\{x : x \text{ is a real number and } x^2 = 1\}.$$

If a set is simple enough (e.g., finite), we sometimes denote it by listing its elements inside curly brackets. For example, the set above can also be denoted by

$$\{-1, 1\}.$$

**1.17 DEFINITION.** Let $A$ and $B$ be sets.

a) $A$ is said to be a *subset* of $B$ or *contained in* $B$ (notation: $A \subseteq B$) if and only if every element of $A$ is an element of $B$. (Thus $\emptyset$ is a subset of ALL sets.)

b) $A$ is said to be a *proper subset* of $B$ or *properly contained in* $B$ (notation: $A \subset B$) if and only if $A \subseteq B$ and there is at least one element of $B$ which does not belong to $A$.

For example, $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$.

The symbols $\subseteq$ and $\subset$ are analogues of the symbols $\leq$ and $<$. The latter implies the former but not conversely. Thus $\{1,2\} \subseteq \{1,2,3\}$, $\{1,2,3\} \subseteq \{3,2,1\}$, and $\{1,2\} \subset \{1,2,3\}$ are all true, but $\{1,2,3\} \subset \{3,2,1\}$ is not. In fact $A \subseteq A$ always holds no matter what set $A$ is.

**1.18 Intervals.** A subset of $\mathbf{R}$ is called an *interval* if it is one of the following four types (the symbol $:=$ means equal by notation or definition):

$$\begin{aligned} (a,b) &:= \{x \in \mathbf{R} : a < x < b\}, \qquad a < b, \\ [a,b) &:= \{x \in \mathbf{R} : a \leq x < b\}, \qquad a < b, \\ (a,b] &:= \{x \in \mathbf{R} : a < x \leq b\}, \qquad a < b, \\ [a,b] &:= \{x \in \mathbf{R} : a \leq x \leq b\}, \qquad a \leq b. \end{aligned}$$

Thus $0 \in [0,1)$, $1/2 \in (0,1)$, $0 \notin (0,1)$, and $2 \notin (0,1)$.

**1.19 Examples.** *Prove the following statements*

a) $(-1,1] \subseteq [-1,1]$ b) $[0,1) \subset [0,1]$ c) $[0,1)$ *is not a subset of* $(0,1]$.

PROOF. a) According to Definition 1.17, to show that $(-1,1]$ is a subset of $[-1,1]$, we must show that each element which belongs to the former belongs to the latter. Let $x \in (-1,1]$. By the definition of intervals, $-1 < x \leq 1$. In particular (that is being less specific), $-1 \leq x \leq 1$, i.e., $x \in [-1,1]$.

b) To show that $[0,1)$ is a proper subset of $[0,1]$, we must show $[0,1) \subseteq [0,1]$ but that the latter contains at least one element which does not belong to the former. Let $x \in [0,1)$. By the definition of intervals, $0 \le x < 1$. In particular, $0 \le x \le 1$, i.e., $x \in [0,1]$. This shows that $[0,1) \subseteq [0,1]$. To finish this part of the proof, notice that $1 \in [0,1]$ but $1 \notin [0,1)$.

c) To show that $[0,1)$ is not a subset of $(0,1]$ all we have to do is show that there is one element of the former which does not belong to the latter. This is easy: $0 \in [0,1)$ but $0 \notin (0,1]$). ∎

**1.20 DEFINITION.** Two sets $A$ and $B$ are said to be equal if and only if $A \subseteq B$ and $B \subseteq A$.

Note: This definition shows that the way to prove that two sets are equal is to show each is a subset of the other.

**1.21 Example.** *Prove that*

$$\{x \in \mathbf{R} : 0 \le 3x < 9\} = [0,3).$$

PROOF. "$\subseteq$" Let $x \in \{x \in \mathbf{R} : 0 \le 3x < 9\}$ i.e., $0 \le 3x < 9$. Recall that when $a < b$ is multiplied by a number $d$, we have $ad < bd$ when $d$ is positive and $ad > bd$ when $d$ is negative. Since $d = 1/3$ is positive, multiplying $0 \le 3x < 9$ by $d$ results in $0 \le x < 3$. Thus $x \in [0,3)$.

"$\supseteq$" Let $x \in [0,3)$, i.e., $0 \le x < 3$. Multiplying this inequality by 3 yields $0 \le 3x < 9$. Thus $x \in \{x \in \mathbf{R} : 0 \le 3x < 9\}$. ∎

**1.22 The outline of nearly every direct proof.** In previous mathematics courses, you were often given rote procedures to follow for every major problem type. For example, if you were asked to find the local extrema of a function, you knew exactly what to do: Take the derivative, find all critical numbers, and then check each of these individually using either the first or second derivative tests. You will be given no such rote procedures for this course. This will probably make you uneasy at first, but let's face it. If what you do in your eventual job can be reduced to an algorithm, sooner or later they will replace your job by a computer. You will survive the vicissitudes of the increasingly volatile job market only if you can think. In particular, don't search the examples in a given section for a template to solve each of the exercises in that section. No such template will be given.

However, don't get too worried. Although proof construction cannot be reduced to rote procedures, there are basic steps common to all direct proofs.

Step 1 *Suppose the hypothesis.* This may seem like a no-brainer, but you'd be surprised how many students fail to recognize that this crucial step is the beginning of all direct proofs.

Step 2 *Use the definition to interpret the hypothesis.* This step is also fairly routine. Whatever the hypothesis assumes must mean something in plain, ordinary language. Look up the definition (if you haven't already memorized it) and write out what that hypothesis means.

Step 3 *Do something creative.* This is the hardest part. You might use algebra. You might break the argument into cases: If the hypothesis implies "$A$" or "$B$," you might use $A$ for Case 1 and $B$ for Case 2; if "$C$" figures prominently in the conclusion, you might use "$C$" for Case 1, and "not $C$" for Case 2.

If you're still having trouble with this step, write out what the conclusion means in plain, ordinary language, and start working backwards trying to reach the hypothesis. The one thing you don't want to do is just give up and ask someone else to show you how to prove it. The struggle you endure trying to solve these homework problems will strengthen you. You will be amazed at how easy some of this stuff is by the end of the semester, **if you work at it!**

Step 4 *Verify the conclusion.* This is usually done directly from Step 3, but if cases are involved, many times one of the cases IS the conclusion, and you must show that the other case is extraneous.

Indirect proofs involve much the same outline, except that you have both the hypothesis and "not the conclusion" to interpret in Step 2, and Step 4 becomes *obtain a contradiction.*

**1.23 DEFINITION.** Let $A$ and $B$ be sets.

a) The *complement* of $B$ relative to $A$ (notation: $A \setminus B$) is the set of all elements $x$ such that $x \in A$ and $x \notin B$.
b) If $B$ is a subset of $\mathbf{R}$, the *complement* of $B$ (notation: $B^c$) is the set $\mathbf{R} \setminus B$.

**1.24 Examples.** *If* $A = [0,1)$, $B = (0,1)$, *and* $C = [-1,2]$, *then*

$$\begin{aligned} A \setminus B &= \{0\} \\ A \setminus C &= \emptyset \\ B \setminus A &= \emptyset \\ C \setminus A &= \{x : -1 \le x < 0 \text{ or } 1 \le x \le 2\} \\ C \setminus B &= \{x : -1 \le x \le 0 \text{ or } 1 \le x \le 2\} \\ A^c &= \{x : x < 0 \text{ or } x \ge 1\} \\ C^c &= \{x : x < -1 \text{ or } x > 2\}. \end{aligned}$$

**1.25 DEFINITION.** Let $A$ be a subset of $\mathbf{R}$.

a) The *reflection* of $A$ is defined to be the set

$$-A := \{x \in \mathbf{R} : -x \in A\}.$$

b) A set $A \subseteq \mathbf{R}$ is said to be *symmetric* if and only if $-A = A$.

For example, $-[0,1) = (-1,0]$, so $[0,1)$ is not symmetric. On the other hand, $-(-1,1) = (-1,1)$, so $(-1,1)$ is symmetric.

You may find the following result useful when working with reflections.