Homework Set # 8 SOLUTIONS – Math 371 – Fall 2009
Quiz Date: 11/24

1. Convert the problem

$$y'' - 0.1(1 - y^2)y' + y = 0$$

with $y(0) = 1$ and $y'(0) = 1$ to a first order system.

Solution:

Let $y_2 = y'$, then the ODE becomes $y_2' - 0.1(1 - y^2)y_2 + y = 0$ and the inital conditions become $y(0) = 1$ and $y_2(0) = 1$. We could also write this as

$$z' = \begin{bmatrix} y' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ 0.1(1 - y^2)y_2 - y \end{bmatrix}$$

such that $z(0) =< 1, 1 >$.

2. Suppose that

$$\frac{dy}{dt} = f(t, y(t)) \ .$$

(a) Use a Taylor Series expansion with a remainder term to show that

$$y(t_n + h) = y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2}\frac{df}{dt}(t_n, y(t_n)) + \frac{h^3}{6}\frac{d^2 f}{dt^2}(\xi, y(\xi)) \ ,$$

where $\xi \in (t_n, t_n + h)$.

(b) Write an algorithm for generating approximates $y_0, y_1, y_2, \ldots$ using the expansion from part (a). Show that it has local truncation error of order h^3.

(c) Apply this method to the ODE (method is referred to as Taylor's method of order 2)

$$\frac{dy}{dt} = t + y$$

with $y(0) = 0$.

Solution:

(a) The general taylor polynomial of order 2 with remainder for y is

$$y(t) = y(c) + y'(c)(t - c) + \frac{1}{2}y''(c)(t - c)^2 + \frac{1}{6}y^{(3)}(z)(t - c)^3$$

for some z between t and c. If we let the basepoint $c = t_n$, recognize that $y'(t) = f(t, y(t))$ and evaluate the Taylor polynomial with remainder at $t = t_n + h$, we get

$$y(t_n + h) = y(t_n) + hf(t_n, y(t_n)) + \frac{1}{2}f'(t_n, y(t_n))h^2 + \frac{1}{6}f''(z, y(z))h^3$$

for some $z \in (t_n, t_n + h)$. and we are done.

(b) We can get a numerical method of order 2 by letting

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2} f'(t_n, y_n) \ .$$

The local truncation error can be obtained by subtracing $y(t_{n+1}) - y_{n+1}$, and assuming that all prior steps are made without error, so that

$$y(t_{n+1}) - y_{n+1} = y(t_n) - y_n + h[f(t_n, y(t_n)) - f(t_n, y_n)] + \frac{h^2}{2}[f'(t_n, y(t_n)) - f'(t_n, y_n)] + \frac{h^3}{6} f''(z) \quad (1)$$

$$= \frac{h^3}{6} f''(z) \quad (2)$$

because we've assumed that $y(t_n) = y_n$. Thus, the l.t.e. is of order h^3, and the method is order 2.

(c) If $y' = t + y$, then $f(t, y) = t + y$, so that our method becomes

$$y_{n+1} = y_n + h(t_n + y_n) + \frac{h^2}{2}(1 + (t_n + y_n))$$

or

$$y_{n+1} = (1 + h + h^2/2) y_n + (h + h^2/2) t_n + h^2/2$$

If $y(0) = 0$, then $y(t_1) = h^2/2$, and $y(t_2) = (1 + h + h^2/2)(h^2/2) + h^2 + h^3/2 + h^2/2 = 2h^2 + h^3 + h^4/4$, and so on...

3. Find the ranges for h that yield stability for the implicit trapezoid method

$$y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1})) \ ,$$

applied to the problem $y' = \lambda y$, with $y(0) = y_0$.

For the ODE $y' = \lambda y$, we have $f(t, y) = \lambda y$, so the method becomes

$$y_{n+1} = y_n + \frac{h}{2}(\lambda y_n + \lambda y_{n+1})$$

or

$$y_{n+1} = \frac{2 + \lambda h}{2 - \lambda h} y_n \ .$$

Starting with $n = 0$, then, we get

$$y_1 = \frac{2 + \lambda h}{2 - \lambda h} y_0$$

$$y_2 = \frac{2 + \lambda h}{2 - \lambda h} y_2 = \left(\frac{2 + \lambda h}{2 - \lambda h}\right)^2 y_0$$

$$y_3 = \left(\frac{2 + \lambda h}{2 - \lambda h}\right)^3 y_0$$

and so on, so that in general

$$y_n = \left(\frac{2 + \lambda h}{2 - \lambda h}\right)^n y_0$$

Thus any error in y_0 gets amplifed by $\left(\frac{2+\lambda h}{2-\lambda h}\right)^n$ at the n-th step. So for stability, we need

$$\left|\frac{2+\lambda h}{2-\lambda h}\right| \leq 1$$

or

$$|2+\lambda h| \leq |2-\lambda h|$$

which gives, by squaring both sides

$$4 + 2\lambda h + \lambda^2 h^2 \leq 4 - 2\lambda h + \lambda^2 h^2$$

or

$$4\lambda h \leq 0$$

Since $h > 0$ is always true, we see that if $\lambda > 0$, the method is unstable for all h and if $\lambda < 0$ the method is stable for all h.

4. Derive a three-step implicit method that is accurate on polynomials up to degree three.

   We can start, as in class, by assuming the form

$$y_{n+1} = \alpha_1 y_n + h(\beta_0 f(t_{n+1}, y_{n+1}) + \beta_1 f(t_n, y_n) + \beta_2 f(t_{n-1}, y_{n-1}) + \beta_3 f(t_{n-2}, y_{n-2}))\ .$$

   Now, we just need to find coefficients $\alpha_1, \beta_0, \beta_1, \beta_2, \beta_3$ that make this method to be order 3. We can do this by making sure that it is exact on any cubic polynomial. First, make sure that it is exact on constatants, so on $y(t) = 1$. Note that then $y'(t) = 0$, so the method becomes

$$1 = \alpha_1 * 1 + 0$$

   so that $\alpha_1 = 1$ must be true. Now, let $y(t) = t$, so that $y'(t) = 1$. Then

$$t_{n+1} = t_n + h(\beta_0 + \beta_1 + \beta_2 + \beta_3)$$

   or after simplifying $1 = \beta_0 + \beta_1 + \beta_2 + \beta_3$. Letting $y(t) = t^2$ so that $y'(t) = 2t$ gives

$$t_{n+1}^2 = t_n^2 + 2h(\beta_0 t_{n+1} + \beta_1 t_n + \beta_2 t_{n-1} + \beta_3 t_{n-2})\ .$$

   Again, simplifying gives

$$t_n + h/2 = \beta_0 t_{n+1} + \beta_1 t_n + \beta_2 t_{n-1} + \beta_3 t_{n-2})\ ,$$

   and

$$(1 - (\beta_0 + \beta_1 + \beta_2 + \beta_3))t_n + (1/2 - (\beta_0 - \beta_2 - 2\beta_3))h = 0\ .$$

   Notice that if we apply the condition we got from the equation for $y(t) = t$ of $\beta_0 + \beta_1 + \beta_2 + \beta_3 = 1$ to this current result, we get

$$1/2 - (\beta_0 - \beta_2 - 2\beta_3) = 0\ .$$

   Finally letting $y(t) = t^3$, then $y'(t) = 3t^2$, so that

$$(t_{n+1})^3 = t_n^3 + 3h(\beta_0 t_{n+1}^2 + \beta_1 t_n^2 + \beta_2 t_{n-1}^2 + \beta_3 t_{n-2}^2)\ .$$

Simplifying gives

$$(3-3(\beta_0+\beta_1+\beta_2+\beta_3))t_n^2+(3-(6\beta_0-6\beta_2-12\beta_3))t_nh+(1-(3\beta_0+3\beta_2+12\beta_3)h^2=0\ .$$

Again, applying the two previous results of $1/2-(\beta_0-\beta_2-2\beta_3)=0$ and $\beta_0+\beta_1+\beta_2+\beta_3=1$, this reduces to

$$1-(3\beta_0+3\beta_2+12\beta_3)=0\ .$$

So, we get a system of 3 equations in the four unknowns β_0, β_1, β_2 and β_3 which we can rewrite as a matrix problem

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 & 1/2 \\ 3 & 0 & 3 & 12 & 1 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1/2 \\ 0 & 0 & 1 & 0 & -1/12 \end{array}\right]$$

So that the general solution looks like

$$<\beta_0,\beta_1,\beta_2,\beta_3>=<5/12,2/3,-1/12,0>+\beta_3<2,-3,0,1>$$

. Note that we get a free parameter in the solution because the system is underdetermined (there are fewer equations than there are coefficients). Thus, we can pick β_3 to be anything we want and this will give us a method of order 3 (note, we should not take $\beta_3=0$ because then the method would be 2-step rather than 3-step). For example, we can let $\beta_3=1/12$, then our method is

$$y_{n+1}=y_n+h(\frac{7}{12}f(t_{n+1},y_{n+1})+\frac{5}{12}f(t_n,y_n)+\frac{-1}{12}f(t_{n-1},y_{n-1})+\frac{1}{12}f(t_{n-2},y_{n-2}))\ .$$

5. Show that the modified Euler Method can be constructed as a predictor corrector method that uses Euler's (explict) method as the predictor, and some implicit method as the corrector (identify which one).

Solution:

Using Euler's method as a predictor means we will first approximate y_{n+1} by

$$y_{n+1}=y_n+hf(t_n,y_n)\ .$$

Now, this gives an approximation for $y'(t_{n+1})$ by using the ODE:

$$y'(t_{n+1})=f(t_{n+1},y(t_{n+1}))\approx f(t_{n+1},y_n+hf(t_n,y_n))$$

If we now use Backward Euler as our corrector, we then get a new approximation for y_{n+1} by

$$y_{n+1}=y_n+hf(t_{n+1},y_{n+1})\approx y_n+hf(t_{n+1},y_n+hf(t_n,y_n))\ .$$

Notice that the final approximation for y_{n+1} is then

$$y_{n+1}=y_n+hf(t_{n+1},y_n+hf(t_n,y_n))\ ,$$

which is exactly what we called the "modified Euler method". So, by using Euler's method as a predictor and backward Euler's method as a corrector we can get the same thing as we get for the modified Euler method.