Homework Set # 7 – Math 371 – Fall 2009
Quiz Date: none! practice for exam 2

1. Consider the integral $\int_1^2 x\cos(\pi x)\,dx$.

   (a) Compute the integral exactly by hand.

   (b) Approximate the integral using the Midpoint rule, Trapezoid rule, Simpson's Rule, composite Simpson's rule, and the extrapolated Simposon's rule. Compute the error for each rule.

   (c) Finally, chop the interval into 2 equal pieces and do Simpson's rule on each piece. By what factor is the error improved? How does the answer compare with using composite Simpson's rule on each piece?

   solutions:

   (a) $\frac{2}{\pi^2} \approx 0.202642367284676$

   (b) $m = 0$, so $E_M = M - true = -\frac{2}{\pi^2} \approx -0.20264$
   $T = .5(2-1) = \frac{1}{2}$, so $E_T = T - true = \frac{1}{2} - \frac{2}{\pi^2} \approx .29736$
   $S = 0 + \frac{1}{6} = \frac{1}{6} \approx 0.16667$, so $E_S \approx -0.035975700618009$
   $S_2 = \frac{\sqrt{2}+1}{6} \approx 0.201184463531091$, so $E_{S_2} \approx -0.001457903753584$
   $Q = S_2 + \frac{S_2 - S}{15} \approx 0.203485649988720$, and $E_Q \approx .0008432827040439905$

   (c) Using simpson's rule on two subinterals give the approximate integral value 0.201184463531091 (this is exactly the same as using the composite simpson's rule over the whole interval). Thus the error for this approximation is -0.001457903753584. The ratio of the error for Simpson's rule over one interval to the error for simpson's rule over 2 subintervals is approximately 24.68. Recall that Simpson's rule is order 4, so by cutting the size of the interval by a factor of 2, we would expect improvement in the error by a factor on the order of $2^4 = 16$ (in fact $24.68 \approx (2.23)^4$, so we can see that this is close)

   If instead we use the composite Simpson's rule on each subinterval (so like doing Simpson's rule on four subintervals), we get the approximate integral value 0.202559533795229. Thus the error for this approximation is $-.00008283348944668201$. The ratio of the error for Simpson's rule over one interval to the error for simpson's rule over 4 subintervals is approximately 434.313. Recall that Simpson's rule is order 4, so by cutting the size of the interval by a factor of 4, we would expect improvement in the error by a factor on the order of $4^4 = 256$ (in fact $434.313 \approx (4.5)^4$, so we can see that this is close).

   Now, comparing the result for composite Simpson's rule on each subinterval and Simpson's rule on each subinterval, we get a ratio of errors 17.600414558446619 which is expected since, again we have halved the size of each interval, which results in a decrease in the error by a factor of about $2^4 = 16$ for an order 4 method. In general, the more times we divide in half, the closer the ratios of the error in successive approximations will be to 16, since the higher order terms in the error are more and more negligible (because the size of the subintervals gets smaller and smaller).

2. Approximate the following integral using your favorite quadrature rule: $\int_1^2 \frac{\cos(\pi x)}{\sqrt{x}}\,dx$. Notice that we cannot integrate this by hand formulaically, so there is no way to check how well we've approximated the integral. How could you ensure that the value you've obtained is fairly accurate? Can you think of a way to estimate the error?

Solutions Using the composite Simpson's rule, we have that the integral is approximately $S_2 = \frac{1}{12} * (f(1) + 4 * f(5/4) + 2 * f(3/2) + 4 * f(7/4) + f(2)) \approx -0.057052117637397$.

How can we estimate the error in this? We can get a (possibly very) rough estimate by noting that

$$E_{S_2} \approx C_1 * (\frac{b-a}{2})^4 = C_1 * (\frac{1}{2})^4$$

since S_2 is order 4 and since the subinterval size for composite simpson's rule is $(b-a)/2 = 1/2$. We can then also note that if we use straight Simpson's rule (which gives an integral approximation of -0.048815536468909), the error is approximately

$$E_S \approx C_1 * (b-a)^4 = C_1 * (1)^4$$

since now we are just using the full interval for the approximation of the integral. We then have the two equations in two unknowns:

$$\begin{aligned} S_2 - \text{ actual } &\approx C_1/16 \\ S - \text{ actual } &\approx C_1 \end{aligned}$$

Solving this for C_1 we have $C_1 \approx 16(S - S2)/15 \approx 0.008786$, which gives that the error for the composite simpsons rule is about $E_{S_2} \approx C_1/16 \approx .0005491054112325505$.

[NOTE: MATLAB's 'quad' function approximates the integral to be -0.057471086850461 using a tolerance of 10^{-8}. This would give us an error of $(-0.057052 + 0.057471) \approx 0.00049$ which is close to our estimate of 0.00055.]

3. Show that the integral of the Hermite interpolating polynomial

$$P_k(s) = \frac{3hs^2 - 2s^3}{h^3} y_{k+1} + \frac{h^3 - 3hs^2 + 2s^3}{h^3} y_k + \frac{s^2(s-h)}{h^2} d_{k+1} + \frac{s(s-h)^2}{h^2} d_k$$

over one subinterval is

$$\int_0^h P_k(s)\, ds = h\frac{y_{k+1} + y_k}{2} - h^2 \frac{d_{k+1} - d_k}{12}$$

Let me know if you want the work for this posted... should be straightforward

4. (by request!) Solve the following matrix problem for the least-squares approximation

$$\begin{pmatrix} 1 & 6 \\ 2 & 9 \\ 2 & 18 \end{pmatrix} \vec{\beta} = \begin{pmatrix} \frac{3}{2} \\ 3 \\ 3 \end{pmatrix}$$

by using appropriate Householder reflections to reduce the matrix to row echelon form and then using back substitution. (do by hand! this one works out nicely)

Solution: This was the best I could do so far in making a problem like this that works out "nicely". Here goes:

To clear the first column under the diagonal, we take H_1 which is defined by the vector

$$u_1 = < 1; 2; 2 > + 3 * < 1; 0; 0 > = < 4, 2, 2 > \quad .$$

Applying H_1 to our matrix gives

$$\begin{pmatrix} -3 & -20 \\ 0 & -4 \\ 0 & 5 \end{pmatrix}$$

and to the right hand side vector $< 3/2; 3; 3 >$ gives

$$\begin{pmatrix} \frac{-9}{2} \\ 0 \\ 0 \end{pmatrix}.$$

Now, to clear the next column under the diagonal, we need H_2 defined by the vector

$$u_2 =< 0; -4; 5 > -\sqrt{41} < 0; 1; 0 >=< 0; -(4 + \sqrt{(}41)); 5 > \ .$$

Applying this to both sides gives the final matrix problem

$$\begin{pmatrix} -3 & -20 \\ 0 & \sqrt{41} \\ 0 & 0 \end{pmatrix} \vec{\beta} = \begin{pmatrix} \frac{-9}{2} \\ 0 \\ 0 \end{pmatrix}$$

We can then solve this via backsubstitution to see that $\beta_2 = 0$ and $\beta_1 = 3/2$.