Homework Set # 6 – Math 371 – Fall 2009
Quiz Date: 10/29/2009

1. The average scores reported by golfers of various handicaps on a difficult par-three hole are as follows:

Handicap	6	8	10	12	14	16	18	20	22	24
Average	3.8	3.7	4.0	3.9	4.3	4.2	4.2	4.4	4.5	4.5

   (a) Construct the matrix problem $X\vec{\beta} = \vec{y}$ associated to this data assuming that you are looking for the least-squares linear approximation. Does this matrix have full rank?

   (b) Find the associated "normal equations" - i.e. find the system $N\vec{\beta} = \vec{z}$, where $N = X^T X$ and $\vec{z} = X^T \vec{y}$. Is this problem well or ill conditioned?

   (c) Construct the matrix problem associated to this data assuming that you are looking for the least-squares quadratic approximation, and check the rank. Then find the associated "normal equations". Again, is this well or ill conditioned?

   (d) Use MATLAB to solve the systems for both the linear and quadratic cases. Explain what exactly you solve, how you solve it, and why you chose that solution method.

   (e) From the results you obtain, do you think that the underlying data set has behavior closer to linear or quadratic? Why?

Solutions:

(a) $$\begin{pmatrix} 6 & 1 \\ 8 & 1 \\ 10 & 1 \\ 12 & 1 \\ 14 & 1 \\ 16 & 1 \\ 18 & 1 \\ 20 & 1 \\ 22 & 1 \\ 24 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 3.8 \\ 3.7 \\ 4.0 \\ 3.9 \\ 4.3 \\ 4.2 \\ 4.2 \\ 4.4 \\ 4.5 \\ 4.5 \end{pmatrix}$$

this matrix has full rank (rank =2 = number of columns)

(b) the normal equations are:

$$\begin{pmatrix} 2580 & 150 \\ 150 & 10 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 637.2 \\ 41.5 \end{pmatrix}$$

Since the condition number for $X^T X$ is approximately 2030.8 under the L^2 norm, this problem is relatively ill-conditioned.

(c) $$\begin{pmatrix} 36 & 6 & 1 \\ 64 & 8 & 1 \\ 100 & 10 & 1 \\ 144 & 12 & 1 \\ 14^2 & 14 & 1 \\ 16^2 & 16 & 1 \\ 18^2 & 18 & 1 \\ 400 & 20 & 1 \\ 22^2 & 22 & 1 \\ 24^2 & 24 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3.8 \\ 3.7 \\ 4.0 \\ 3.9 \\ 4.3 \\ 4.2 \\ 4.2 \\ 4.4 \\ 4.5 \\ 4.5 \end{pmatrix}$$

This matrix also has full rank (rank = 3 = number of columns)

The resulting normal equations are $\begin{pmatrix} 971088 & 48600 & 2580 \\ 48600 & 2580 & 150 \\ 2580 & 150 & 10 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 11144 \\ 637 \\ 42 \end{pmatrix}$

Again, the condition number for this matrix is $510954.2\ldots$ under L^2 norm, so this problem is certainly ill-conditioned.

(d) The solution for the linear problem is $y(t) = 0.04454545t + 3.481818$. The solution for the quadratic problem is $y(t) = -.0004735t^2 + .05875t + 3.39090909$. I solved the original equations with the backslash operator. I avoided the normal equations since they are both ill-conditioned. Both matrices are of full rank, so we have no issues with possible multiple least-squares solutions and there is no need to use the pseudoinverse. [It's good to point out that there is a big difference between the backslash solution of the original quadratic solution and the backslash solution of the normal equations... this is because the normal equations are far more ill-conditioned than the original. recall that cond(X)$^2 = cond(X^T X)$]

(e) I would posit that the underlying data is actually closer to linear. You can see that the coefficient on the quadratic term is very small, so the quadratic least squares fit is actually nearly linear (and the other terms are very close to those of the linear least squares fit).
We could also try looking at the two residuals $\|X\vec{\beta} - \vec{y}\|_2^2$... The residual for the linear fit is $0.0901818\ldots$. The residual for the quadratic fit is $0.088287878\ldots$. You can see that for this particular case, the residual doesn't help us determine which is a better fit, because the quadratic approx is so close to the linear one the error is nearly the same.

2. Suppose H is a Householder reflection. Show that $H^T = H$ and $H^2 = I$.

   Solution: Since $H = I - \rho\vec{u} * \vec{u}^T$ and since $(BA)^T = A^T B^T$ and $(B+A)^T = B^T + A^T$, we see that

$$H^T = (I - \rho\vec{u}\vec{u}^T)^T = I^T - \rho(\vec{u}^T)^T\vec{u}^T = I - \rho\vec{u}\vec{u}^T = H$$

   Also $H^2 = (I - \rho\vec{u} * \vec{u}^T)(I - \rho\vec{u} * \vec{u}^T)$, so

$$H^2 = I - 2\rho\vec{u} * \vec{u}^T + \rho^2\vec{u} * \vec{u}^T\vec{u} * \vec{u}^T = I - 2\rho\vec{u} * \vec{u}^T + \rho^2\vec{u}(\|u\|^2)\vec{u}^T$$

   Thus, using the fact that $\rho = \frac{2}{\|u\|^2}$ we can simplify this further to

$$H^2 = I - 2\rho\vec{u} * \vec{u}^T + 2\rho\vec{u} * \vec{u}^T = I$$

3. Again, suppose that H is a Householder reflection, so that $H = I - \rho\vec{u}\vec{u}^T$. Fix vector $\vec{x} \in \mathbb{R}^n$. Suppose further that u is obtained by

$$\vec{u} = \vec{x} + sign(x_k)||\vec{x}||_2\vec{e}_k \ .$$

   Show that $H\vec{x} = C\vec{e}_k$ for some constant C.

   Solution:

$$H\vec{x} = \vec{x} - \rho\vec{u}\vec{u}^T\vec{x} = \vec{x} - \rho\vec{u}(\vec{x}^T + sign(x_k)\|\vec{x}\|\vec{e}_k^T)\vec{x}$$

   Multiplying out, and using the fact that $\vec{x}^T\vec{x} = \|\vec{x}\|^2$, we get

$$H\vec{x} = \vec{x} - \rho\vec{u}(\|\vec{x}\|^2 + sign(x_k)\|\vec{x}\|x_k)$$

Now, since $\rho = \frac{2}{\|u\|^2} = \frac{2}{\|\vec{x}\|^2 + 2sign(x_k)\|\vec{x}\|x_k + \|\vec{x}\|^2} = \frac{1}{\|\vec{x}\|^2 + sign(x_k)\|\vec{x}\|x_k}$, subbing this in we get

$$H\vec{x} = \vec{x} - \vec{u} = \vec{x} - (\vec{x} + sign(x_k)\|\vec{x}\|\vec{e}_k) = -sign(x_k)\|\vec{x}\|\vec{e}_k$$

4. Let $\vec{x} =< 9; 2; 6 >$. Use problem 3 to find the Householder reflection H that transforms $\vec{x}$ into

$$H\vec{x} = \begin{pmatrix} -11 \\ 0 \\ 0 \end{pmatrix}$$

What Householder reflection H_2 would instead transform $\vec{x}$ into

$$H_2\vec{x} = \begin{pmatrix} 0 \\ -11 \\ 0 \end{pmatrix} ?$$

Solution:

Notice that $\| < 9, 2, 6 > \| = 11$. So, by number 3, we can take H to be the householder reflection where $\vec{u} =< 9, 2, 6 > +(1)(11) < 1, 0, 0 >=< 20, 2, 6 >$. Then

$$H = I - \frac{2}{440} < 20; 2; 6 > * < 20, 2, 6 >= I - \frac{1}{220} \begin{pmatrix} 400 & 40 & 120 \\ 40 & 4 & 12 \\ 120 & 12 & 36 \end{pmatrix}$$

So that combining everything into one matrix, we get:

$$H = \begin{pmatrix} -180/220 & -40/220 & -120/220 \\ -40/220 & 216/220 & -12/220 \\ -120/220 & -12/220 & 184/220 \end{pmatrix}$$

Check:

$$H\vec{x} = \begin{pmatrix} -180/220 & -40/220 & -120/220 \\ -40/220 & 216/220 & -12/220 \\ -120/220 & -12/220 & 184/220 \end{pmatrix} \begin{pmatrix} 9 \\ 2 \\ 6 \end{pmatrix}$$

implies $H\vec{x} =< -2420/220, 0, 0 >=< -121/11, 0, 0 >=< -11, 0, 0 >$:).

In order to get $H\vec{x} =< 0, -11, 0 >$, we could take the Householder reflection defined by $\vec{u} =< 9; 2; 6 > +11 < 0, 1, 0 >=< 9, 13, 6 >$.

Check: $H = I - \frac{2}{286} \begin{pmatrix} 81 & 117 & 54 \\ 117 & 169 & 78 \\ 54 & 78 & 36 \end{pmatrix}$ gives

$$H = \begin{pmatrix} 62/143 & -117/143 & -54/143 \\ -117/143 & -26/143 & -78/143 \\ -54/143 & -78/143 & 107/143 \end{pmatrix}$$

Thus,

$$H\vec{x} =< 0, -1573/143, 0 >=< 0, -11, 0 >$$

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