Homework Set # 4 – Math 371 – Fall 2009
No Quiz! Will be included in exam on 9/29

1. Suppose you have the data set $\{(1,10),(2,15),(4,8),(5,2),(7,12)\}$. Use this data for the following parts:

   (a) Find the full interpolation polynomial of degree 4 determined by these five points.

   (b) Find the piecewise linear interpolation polynomial for these five points.

   (c) Find the shape-preserving piecewise cubic polynomial that fits these five points, with $d_1 = d_5 = 0$ end conditions.

   (d) Find the cubic spline piecewise polynomial, with "not a knot" spline end conditions for these five points.

   Solution:

   (a) Note: the Vandermonde matrix corresponding to this data set is very illconditioned. Better to use the Legendre form for finding the polynomial.

$$\begin{aligned} P(x) &= \frac{10}{72}(x-2)(x-4)(x-5)(x-7) + \frac{15}{-30}(x-1)(x-4)(x-5)(x-7) \\ &= \frac{8}{18}(x-1)(x-2)(x-5)(x-7) + \frac{2}{-24}(x-1)(x-2)(x-4)(x-7) \\ &= \frac{12}{180}(x-1)(x-2)(x-4)(x-5) \end{aligned}$$

   (b) We have 4 intervals between data points, so we have for polynomials. P_k denotes the polynomial on the interval $[x_k, x_{k+1}]$.

$$P_1(x) = 10 + 5(x-1)$$

$$P_2(x) = 15 - \frac{7}{2}(x-2)$$

$$P_3(x) = 8 - 6(x-4)$$

$$P_4(x) = 2 + 5(x-5)$$

   (c) $d_2 = 0$ because the slopes around x_2 have opposite sign. $d_3 = -420/95$ using the weighted harmonic mean given in the text. $d_4 = 0$ again because the slopes around x_4 have opposite sign. this yields the polynomials

$$\begin{aligned} P_1(x) &= 15(3(x-1)^2 - 2(x-1)^3) + 10(1 - 3(x-1)^2 + 2(x-1)^3) \\ P_2(x) &= (6(x-2)^2 - 2(x-2)^3) + \frac{15}{8}(8 - 6(x-2)^2 + 2(x-2)^3) - \frac{420}{95*4}(x-2)^2((x-2)-2) \\ P_3(x) &= 2(3(x-4)^2 - 2(x-4)^3) + 8(1 - 3(x-4)^2 + 2(x-4)^3) - \frac{420}{95}(x-4)((x-4)-1)^2 \\ P_4(x) &= \frac{12}{8}(6(x-5)^2 - 2(x-5)^3) + \frac{2}{8}(8 - 6(x-5)^2 + 2(x-5)^3) \end{aligned}$$

(d) The "not-a-knot" end conditions at the left end give:

$$\frac{3}{2}d_2 + d_1 = \frac{7}{3}\delta_1 + \frac{1}{3}\delta_2$$

and at the right end we have

$$\frac{3}{4}d_5 + \frac{9}{4}d_4 = 2\delta_4 + \delta_3 \ .$$

For each interior "knot" we get an additional constraint. They are

$$\begin{aligned} 2d_1 + 6d_2 + d_3 &= 6\delta_1 + 3\delta_2 \\ d_2 + 6d_3 + 2d_4 &= 3\delta_2 + 6\delta_3 \\ 2d_3 + 6d_4 + d_5 &= 6\delta_3 + 3\delta_4 \end{aligned}$$

Putting all of this into a matrix problem, we have $A\vec{d} = \vec{b}$ where $\vec{d} =< d_1, d_2, d_3, d_4, d_5 >$,

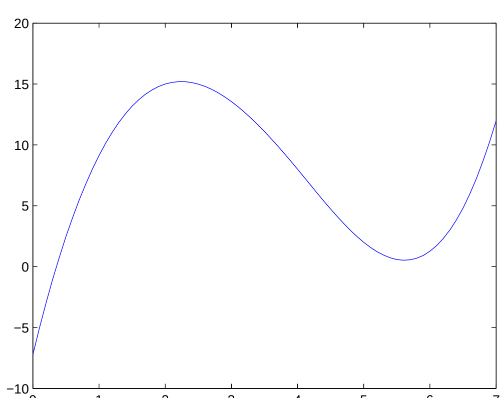
$$A = \begin{bmatrix} 1 & 3/2 & 0 & 0 & 0 \\ 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & 0 \\ 0 & 0 & 2 & 6 & 1 \\ 0 & 0 & 0 & 9/4 & 3/4 \end{bmatrix}$$

and

$$\vec{b} = \begin{bmatrix} \frac{7}{3}\delta_1 + \frac{1}{3}\delta_2 \\ 6\delta_1 + 3\delta_2 \\ 3\delta_2 + 6\delta_3 \\ 6\delta_3 + 3\delta_4 \\ 2\delta_4 + \delta_3 \end{bmatrix}$$

Using the values for the δ_i's found in part (b) and using MATLAB's backslash operator, we have $d_1 = 7.97$, $d_2 = 1.69$ $d_3 = -6.56$, $d_4 = -4.40$, $d_5 = 18.54$ accurate to two decimal places. This gives us the cubic spline polynomials

$$\begin{aligned} P_2(x) &= 15 + 7.97(x-2) + -3.66(x-2)^2 + 0.531(x-2)^3 \\ P_3(x) &= 8 - 6.56(x-4) - 0.47(x-4)^2 + 1.03(x-4)^3 \\ P_4(x) &= 2 - 4.4(x-5) + 2.63(x-5)^2 + 1.03(x-5)^3 \end{aligned}$$



2. Now consider a set of 5 generic coordinates $\{(x_i, y_i)|i = 1, \ldots, 5\}$, with $h_k = x_{k+1} - x_k = h$ where h is a positive constant. Set up the system of 5 equations for the 5 unknown slopes $d_1, \ldots, d_5$ as a matrix problem $A\vec{d} = \vec{y}$ for

   (a) the "Natural Cubic Spline" - i.e. where $P_1''(x_1) = P_{n-1}''(x_n) = 0$

   (b) the "Clamped Cubic Spline" with $d_0 = d_5 = 0$.

   (Note: you just need to find A and $\vec{y}$, but do not need to solve the system)

Solutions:

(a)

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

and $\vec{y} = [3\delta_1; 3(\delta_1 + \delta_2); 3(\delta_2 + \delta_3); 3(\delta_3 + \delta_4); 3\delta_4]$

(b)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\vec{y} = [0; 3(\delta_1 + \delta_2); 3(\delta_2 + \delta_3); 3(\delta_3 + \delta_4); 0]$