Homework Set # 1 – Math 371 – Fall 2009 Quiz date: 9/1/2009

1. If a is an approximate value for a quantity whose true value is t, and a has relative error r, prove from the definitions of these terms that $a = t(1+r)$.

2. Let $p(x) = 1.01x^4 - 4.62x^3 - 3.11x^2 + 12.2x - 1.99$.

   (a) Show that the above polynomial can also be written in nested form as was demonstrated in lecture.

   (b) Use three-digit rounding arithmetic to evaluate p(4.62), using the form of the polynomial given above.

   (c) Use three-digit rounding arithmetic to evaluate p(4.62), using the nested form of the polynomial from part (a).

   (d) Compare the approximations in parts (b) and (c) to the true three-digit result p(4.62) = -7.61 by finding the absolute and relative errors.

3. Consider $p(x) = x^2 + 62.10x + 1 = 0$. The roots of this equation are $x_1 = -0.1610723$ and $x_2 = -62.08390$ to seven significant digits. Use four digit rounding arithmetic to approximate the roots of p using the quadratic formula. Then compute the relative error for each approximated root. One root should be much more accurate than the other - explain why.

4. For computing the midpoint m of an interval $[a, b]$, which of the following two formulas is preferable in floating-point arithmetic? When? Why? (Give examples to illustrate your reasoning)

   (a) $m = (a + b)/2.0$

   (b) $m = a + (b - a)/2.0$

5. Let $\vec{x} \in \mathbb{R}^n$. Show that

$$\|\vec{x}\|_\infty \leq \|\vec{x}\|_2 \leq \sqrt{n}\|\vec{x}\|_\infty \ .$$

Solution:

By the definition of the sup norm, we have $\|\vec{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$. This maximum value is achieved for some index i, let's say for $i = I$. Then $\|\vec{x}\|_\infty = \max_{1\leq i\leq n} |x_i| = |x_I|$. By our choice of I, then, $|x_i| \leq |x_I|$ for all $i \in [1, \ldots, n] \subset \mathbb{Z}$.

Now, since $\sum_{i\neq I}(x_i)^2 \geq 0$, we can add x_I^2 to both sides to see

$$\sum_{1\leq i\leq n} (x_i)^2 \geq (x_I)^2$$

Taking the square root of both sides gives

$$\|\vec{x}\|_2 \geq |x_I| = \|\vec{x}\|_\infty \ .$$

To get the other inequality, we again use the fact that $|x_i| \leq |x_I|$ for all i. This tells us that $x_i^2 \leq x_I^2$ for all i, so that

$$\|\vec{x}\|_2 = \sqrt{\sum_{1\leq i\leq n} x_i^2} \leq \sqrt{\sum_{1\leq i\leq n} x_I^2} = |x_I|\sqrt{\sum_{1\leq i\leq n} 1} = \sqrt{n}\|\vec{x}\|_\infty$$

.

6. Suppose that $\|\cdot\|$ defines a vector norm on $\mathbb{R}^n$. Let A be an $n \times n$ invertible matrix with real entries. Show that

$$\|\vec{x}\|_* = \|A\vec{x}\|$$

defines a norm on $\mathbb{R}^n$.

1. If $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} \in \mathbb{R}^n$ as well. We know that $\|\cdot\|$ is a norm, so by the first property of norms, $\|A\vec{x}\| \geq 0$ must be true. Thus, for any $\vec{x} \in \mathbb{R}^n$,

$$\|\vec{x}\|_* = \|A\vec{x}\| \geq 0 \ ,$$

and $\|\cdot\|_*$ also satisfies the first property of norms.

2. If $\vec{x} = \vec{0}$, then $A\vec{x} = \vec{0}$. By the second norm property for $\|\cdot\|$, $\|\vec{x}\|_* = \|A\vec{x}\| = \|\vec{0}\| = 0$ must be true. So **if** $\vec{x} = \vec{0}$, then $\|\vec{x}\|_* = 0$.

Now, if we assume that $\|\vec{x}\|_* = 0$, this says that $\|A\vec{x}\| = 0$ by the definition of $\|\cdot\|_*$. Since $\|\cdot\|$ is a norm, we know the only way this can be true is if $A\vec{x} = \vec{0}$. Now, since A is invertible, $A\vec{x} = \vec{0}$ if and only if $\vec{x} = \vec{0}$. Thus we have shown that $\|\vec{x}\|_* = 0$ if and only if $\vec{x} = \vec{0}$.

3. $\|\alpha\vec{x}\|_* = \|A(\alpha\vec{x})\|$ by the definition of $\|\cdot\|_*$. Since A is a matrix, we have

$$\|A(\alpha\vec{x})\| = \|\alpha A(\vec{x})\| = |\alpha|\|A\vec{x}\| = |\alpha|\|\vec{x}\|_*$$

where the second to last equality is true because $\|\cdot\|$ is a norm, and the last is true by the definition of $\|\cdot\|_*$. Putting it all together, we see that

$$\|\alpha\vec{x}\|_* = |\alpha|\|\vec{x}\|_* \ .$$

4. Finally, we need to show that the triangle inequality holds for $\|\cdot\|_*$:

$$\|\vec{x} + \vec{y}\|_* = \|A(\vec{x} + \vec{y})\| = \|A\vec{x} + A\vec{y}\|$$

Since $\|\cdot\|$ is a norm, it satisfies the triangle in equality, and we get

$$\leq \|A\vec{x}\| + \|A\vec{y}\| = \|\vec{x}\|_* + \|\vec{y}\|_*$$

Again, putting it all together, we see that

$$\|\vec{x} + \vec{y}\|_* \leq \|\vec{x}\|_* + \|\vec{y}\|_*$$

Thus, $\|\cdot\|_*$ satisfies all the properties of a norm, and so IS a norm.

7. Show that for A defined by

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

for any real numbers a, b, $\|A\|_1 = \|A\|_2 = \|A\|_\infty$.

$\|A\|_1$ = maximum absolute column sum = $|a| + |b|$.

$\|A\|_{\text{inf}}$ = maximum absolute row sum = $|a| + |b|$.

$\|A\|_2 = \sqrt{\rho(A^T A)}$. Let's compute this spectral radius $\rho(A^T A)$.

$$A^T A = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$$

The eigenvalues are values of λ such that $det(\lambda I - A^T A) = 0$. Since

$$\lambda I - A^T A = \begin{bmatrix} \lambda - (a^2+b^2) & -2ab \\ -2ab & \lambda - (a^2+b^2) \end{bmatrix}$$

we have $det(\lambda I - A^T A) = (\lambda - (a^2+b^2))^2 - 4a^2b^2 = 0$, or

$$\lambda = a^2 + b^2 \pm 2|a||b| = (|a| \pm |b|)^2$$

.

Thus, the largest value λ can have is $(|a|+|b|)^2$, so

$$\|A\|_2 = \sqrt{\rho(A^T A)} = |a| + |b| \ ,$$

and all of the norm values are the same.

8. Show that if

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

for any real numbers a, b, then $\|A\|_2 = \sqrt{a^2+b^2}$.

By definition, $\|A\|_2 = \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}$. Using this particular A, we get

$$\|A\|_2 = \max_{\|\vec{x}\| \neq 0} \frac{\sqrt{(ax+by)^2 + (bx-ay)^2}}{\sqrt{x^2+y^2}} = \max_{\|\vec{x}\| \neq 0} \frac{\sqrt{a^2x^2 + 2abxy + b^2y^2 + b^2x^2 - 2abxy + a^2y^2}}{\sqrt{x^2+y^2}}$$

Simplifying, we see

$$= \max_{\|\vec{x}\| \neq 0} \frac{\sqrt{(a^2+b^2)(x^2+y^2)}}{\sqrt{x^2+y^2}} = \max_{\|\vec{x}\| \neq 0} \sqrt{(a^2+b^2)} = \sqrt{a^2+b^2} \ .$$