Solutions to homework – Section 3.3:

1. He showed that the real numbers and the natural numbers have different cardinalities, so that there is more than one "size" of infinity.

2. 0.<u>1</u>2345 ← $1^{st}$ digit is 1
   0.2<u>4</u>242 ← $2^{nd}$ digit is 4
   0.98<u>7</u>65 ← $3^{rd}$ digit is 7

   (by "first digit", generally mean "first <u>nonzero</u> digit")

3. It's constructed by listing all the natural numbers consecutively.

   0.12345678910111213141516171819 20...

   10th  14th  25th  31st

4. Any 5-digit number whose first digit is not 3, second is not 8, and third digit is not 2 will definately not be in the list → for example, 49311 cannot be listed.

5. No, you cannot, because no matter what number I give you, you cannot know whether it's equal to the last number on the list or not.

9. Suppose we have a listing of all the real numbers, as Cantor did. We are going to write down a number, called M, that is missing from this list. We can suppose that $M = 0.???\ldots$ (so it is some number between 0 and 1). Each digit of M will either be a 4 or an 8, and we'll decide which one each digit is by the following:

   For the $1^{st}$ digit of M, we look at the number in the list that is <u>first</u>. Either it is a 4 or it is not a 4. If the $1^{st}$ digit of the $1^{st}$ # of the list <u>is</u> a 4, then we make the $1^{st}$ digit of M to be 8.

If it is not 4, we set the first digit of M to be 4. Now, for the second digit of M, we look at the 2nd digit of the 2nd number in the list. Again, if it's 4, then the 2nd digit of M is set to 8, and if it's not 4, then the second digit of M is 4. We continue on in this way, so that the $n^{th}$ digit of M is 4 if the $n^{th}$ digit of the $n^{th}$ number in the list is not 4, and is 8 if the $n^{th}$ digit of the $n^{th}$ number in the list is 4.

By the way we have created M, it cannot be equal to any number in the list! We can do this process regardless of what list we begin with, so the only conclusion is that there is no possible way to list all of the real numbers. This tells us that there is no one-to-one correspondence between the real numbers + the natural numbers, since if there was a 1-1 corresp., we would then be able to list all the reals.

11. diagonalization is a nice name for the process of Cantor's proof because as you construct the number M that's missing from the list, you only need look at the 1st digit of the 1st number, 2nd digit of the 2nd number, etc, which creates a diagonal in the list. For example:

$1 \leftrightarrow 0.5967784123\ldots$
$2 \leftrightarrow 0.4125689712\ldots$
$3 \leftrightarrow 0.5432142612\ldots$
$4 \leftrightarrow 0.0041002100\ldots$
$\vdots$

14. Each flip gives some infinite sequence of H's and T's

for example:

| Person: | 1 | 2 | 3 | 4 | 5 | 6 | .... |
|---|---|---|---|---|---|---|---|
| Result: | H | T | H | H | H | T | .... |

So the set of all possible outcomes would be the set of all possible infinite lists of H's and T's.

We could show that there is no 1-1 correspondence between the set of all outcomes & $\mathbb{N}$ by the same diagonalization technique Cantor used:

Suppose there is a 1-1 correspondence

$1 \leftrightarrow$ H T H H H T H ...

$2 \leftrightarrow$ H T T H H T T H ...

$3 \leftrightarrow$ T H T H T H H T ...

$4 \leftrightarrow$ T H T T T H T ...

⋮

Then we can always create an outcome that's missed by changing the entry on the diagonal:

missed outcome = T H H H ...

This contradicts the claim that we had a 1-1 correspondence.

16. Show set of all real numbers between 0 and 1 just having 1's and 2's after the decimal has a greater cardinality than the natural numbers ($\mathbb{N}$).

Again, we can show this by contradiction.
Assume they have the <u>same</u> cardinality. then I can list <u>all</u> the numbers between 0 and 1 whose digits are only 1's and 2's. For example:

$$1 \leftrightarrow 0.\underline{1}2211112121\ldots$$
$$2 \leftrightarrow 0.1\underline{2}1212121\ldots$$
$$3 \leftrightarrow 0.11\underline{1}1222212111\ldots$$
$$4 \leftrightarrow 0.122\underline{2}11121211112211\ldots$$
$$\vdots$$

Now, I can always construct <u>another</u> number between 0 and 1 whose digits are only 1's and 2's by again using diagonalization: If the $n^{th}$ digit of the $n^{th}$ number in the list is a 1, I can make the $n^{th}$ digit of the missing number a 2 and vice-versa
So for the list above, my missing member is

$$m = 0.2121\ldots$$

This contradicts the claim that I had a 1-1 correspondence with $\mathbb{N}$.
<u>So</u>, $\mathbb{N}$ and the set of all reals between 0 and 1 with digits only 1's and 2's do <u>not</u> have the same cardinality.

19. Since the Hotel Cardinality only has as many rooms as there are natural numbers, if we had a group of people whose cardinality was the same as the real numbers, there would be no way the Hotel manager could give everyone a room, because there is no 1-1 corresp. between $\mathbb{N}$ (nat. numbers) and $\mathbb{R}$ (real numbers).

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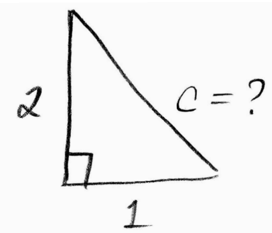
# 4.1 - Solutions

1 - See text

2.

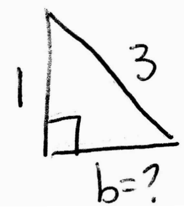



→ Pythag. Thm says $2^2 + 1^2 = c^2$

so $4 + 1 = c^2$

$\Rightarrow 5 = c^2 \Rightarrow \underline{c = \sqrt{5}}.$

If instead:



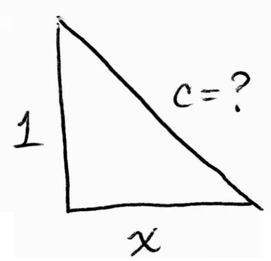

The Pythag. thm says: $1^2 + b^2 = 3^2$

$\Rightarrow 1 + b^2 = 9$

$b^2 = 8$

$\Rightarrow b = \sqrt{8}$

3.

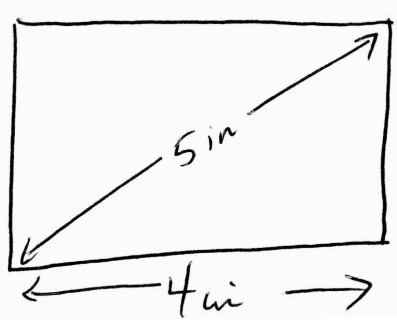



$\Rightarrow 1^2 + x^2 = c^2$ (want to find c!)

$\Rightarrow 1 + x^2 = c^2$

$\Rightarrow c = \sqrt{1 + x^2}$ is the length of the hypotenuse.

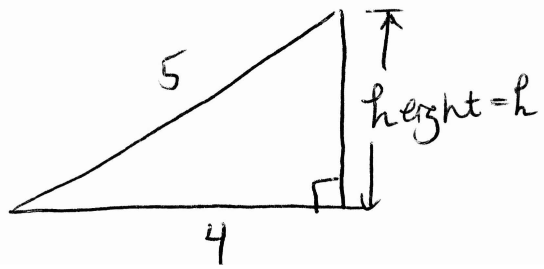
4.




area?

Since Area = base x height, we need to find the height of the rectangle.

we can use Pyth. Thm. on the <u>triangle</u>:




so $4^2 + h^2 = 5^2$

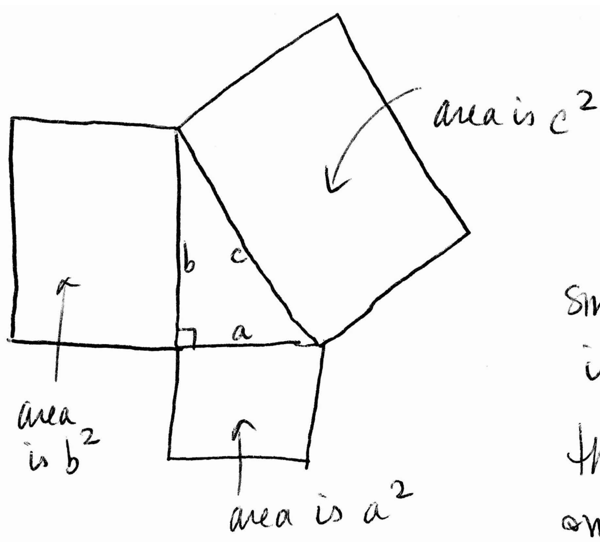
$\Rightarrow 16 + h^2 = 25$

$\Rightarrow h^2 = 9$

$\Rightarrow \boxed{h = 3}$

Since the base = 4 in and height = 3 in, the area is $A = 4 \cdot 3 = 12$ in$^2$.

5.



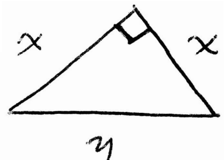

Pythag. Thm says

$a^2 + b^2 = c^2$

Since the triangle in the middle is a right triangle, we know that the area of the square on the hypotenuse is equal to the sum of the areas of the other two squares.

6. (we did in class!) The angles combine to give a straight line $\Rightarrow$ Sum is $180°$.

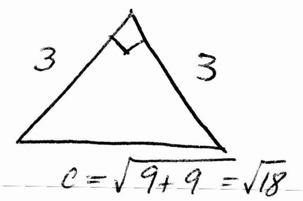

12. Scarecrow said




$\Rightarrow \sqrt{x} + \sqrt{x} = \sqrt{y}$

and $\sqrt{x} + \sqrt{y} = \sqrt{x}$

if the legs have length 3, he is claiming that




Scarecrow says $\longrightarrow$

$$\underbrace{\sqrt{3} + \sqrt{3}}_{= 2\sqrt{3}} \stackrel{?}{=} \sqrt{\sqrt{18}}$$

$\boxed{\text{not true!}}$

(since $(2\sqrt{3})^2 \neq (\sqrt{\sqrt{18}})^2$
$\quad\;\; \| \qquad\qquad\quad \|$
$\quad\; 12 \qquad\qquad \sqrt{18}$ )

pythag. thm says:

$3^2 + 3^2 = c^2$

$\Downarrow$

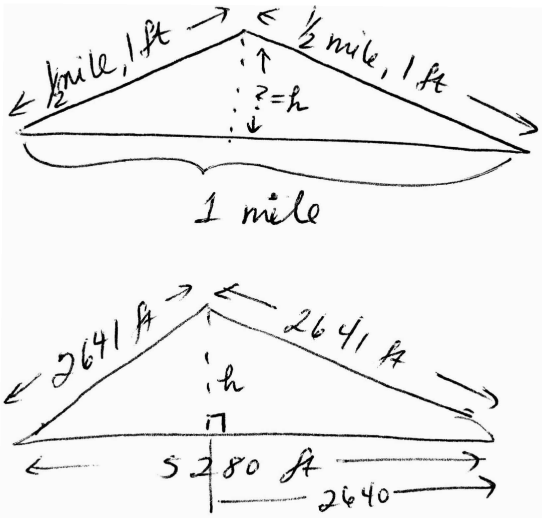
$9 + 9 = c^2$

$\Downarrow$

$c = \sqrt{9+9} = \sqrt{18}$

☆ No, his assertion is not valid.

15.

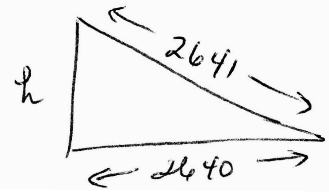



<u>What is h?</u>

Since 1 mile = 5280 ft.

then ½ mile + 1 ft = 2641 ft.

we can look at the right triangle:




$$\Rightarrow h^2 + (2640)^2 = (2641)^2$$

$$\Rightarrow h^2 = 5281$$

$$\Rightarrow h = \sqrt{5281} = 72.67 \text{ feet!}$$

The midpoint would be 72.67 ft high!

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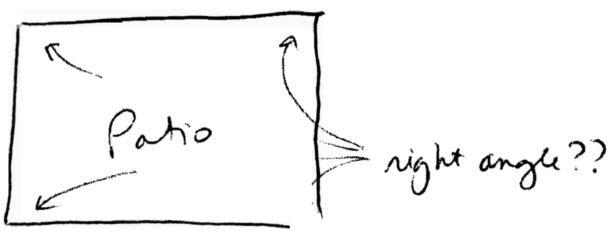
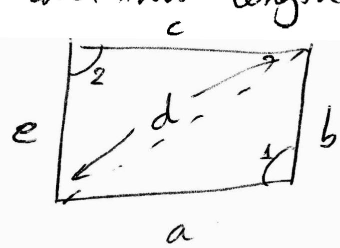
18.




To check ~~each~~ angle, we can first measure one of the diagonals — call this length d:




To check the other 2 angles, you can use the opposite diagonal.

if $a^2 + b^2 = d^2$, then angle 1 is a right angle.

and if $e^2 + c^2 = d^2$, then angle 2 is a right angle!