Answers to Various Problems, Exam 4 Set

Section 12.6 #23 The sphere $x^2 + y^2 + z^2 = 4z$ can be rewritten by

$$x^2 + y^2 + z^2 - 4z = 0$$
$$x^2 + y^2 + z^2 - 4z + 4 = 4$$
$$x^2 + y^2 + (z - 2)^2 = 4$$

so it has center $(0, 0, 2)$ and radius 2. We can parameterize this by $\vec{r}(\theta, \phi) =< 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi + 2 >$, or by $\vec{r}(u, v) =< u, v, 2 + \sqrt{4 - u^2 - v^2} >$ where we chose the positive square root because the part of the sphere we end up with is in the top half. Using the second parametrization: $\vec{r}_u =< 1, 0, \frac{-u}{\sqrt{4-u^2-v^2}} > \vec{r}_v =< 0, 1, \frac{-v}{\sqrt{4-u^2-v^2}} > \vec{r}_u \times \vec{r}_v =< \frac{u}{\sqrt{4-u^2-v^2}}, \frac{v}{\sqrt{4-u^2-v^2}}, 1 >$ so the surface area is equal to

$$\int\int \sqrt{1 + \frac{u^2}{4 - u^2 - v^2} + \frac{v^2}{4 - u^2 - v^2}}\, du\, dv = \int\int \frac{2}{\sqrt{4 - u^2 - v^2}}\, du\, dv =$$

We can change to polar coordinates to finish this problem, but we also need to determine the limits of integration. If you look at the picture of these surfaces posted on the website, you can see that the two surfaces intersect in a circe and the part of the sphere that you want the area of lies inside that circle of intersection. We need to know the radius of this circle of intersection to get our limits of integration. So, if $z = x^2 + y^2$, and $x^2 + y^2 + z^2 = 4z$, then the intersection is given by $z + z^2 = 4z$, or $z^2 = 3z$. This implies that $z^2 - 3z = 0$, so they intersect at $z = 0$ and $z = 3$. The intersection of $z = 3$ is the one we are interested in, and when $z = 3$, the equation of the parabolic surface is $3 = x^2 + y^2$, so we have a circle of radius $\sqrt{3}$. Finally changing to polar, we have

$$\int_0^{2\pi} \int_0^3 \frac{2}{\sqrt{4 - r^2}} r\, dr\, d\theta$$

Now let $u = 4 - r^2$ as a substitution to finish the integral.

#9 When you set up this problem, you can let

$$\vec{r}(u,v) =< u, 4u+v^2, v >$$

, and if you work out the cross product of the partial derivatives and simplify, you see that the surface area is

$$\int_0^1 \int_0^1 \sqrt{17+4v^2}\, du\, dv$$

This integral requires trig substitution - let $v = \frac{\sqrt{17}}{2}\tan\theta$. Feel free to use an integral table on this problem instead.

Section 13.2 #13 Evaluate $\int_C (x+yz)dx + 2x\,dy + xyz\,dz$, where C is the line segments from $(1,0,1)$ to $(2,3,1)$ and then from $(2,3,1)$ to $(2,5,2)$. We can parametrize the two pieces of C. Let's call the first piece C_1, and so C_1 is given by $\vec{r}_1(t) =< 1,3,0 > t+ < 1,0,1 >=< t+1, 3t, 1 >$ with $0 \le t \le 1$. The second piece C_2 is given by $\vec{r}_2(t) =< 2, 2t+3, t+1 >$ again with $0 \le t \le 1$. Setting up the line integral, then, we have

$$\int_0^1 < t{+}1{+}3t, 2t{+}2, (t{+}1)(3t) > \cdot < 1,3,0 > \, dt + \int_0^1 < 2{+}(2t{+}3)(t{+}1), 4, 2(2t{+}3)(t{+}1) > \cdot$$

simplifying, we have

$$\int_0^1 10t + 7\,dt + \int_0^1 8 + 4t^2 + 10t + 6\,dt$$

You can do the rest!