Homework Set # 8 – Math 435 – SOLUTIONS Due date: 3/13/2013

1. (a) Find the Fourier sine series, the Fourier cosine series and the full Fourier series expansion of e^x on $(0,2)$ or $(-2,2)$ as appropriate. (Note that once you've done the work for finding the sine and cosine series coefficients, you need only divide by 2 and change the limits of integration in the integrals used to find the coeffs for the full series - this will save you alot of work!)

   (b) Use MATLAB to plot the approximation by each type of series (for example, using the full series we have $f(x) \approx \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n \cos(n\pi x/l) + B_n \sin(n\pi x/l)))$ for $N = 3, 5, 10, 100$, each one plotted on the same axes along with a plot of the actual function $f(x) = e^x$ (you should have one plot for each type of Fourier series). All of these plots should only be over the interval [-2,2] - and make sure you label each curve. [Let me know if you need some guidance on these MATLAB parts]

Solution:

For the sine series, we want to write

$$e^x = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{2})$$

and we need to find the coefficients A_n that make this equality true.

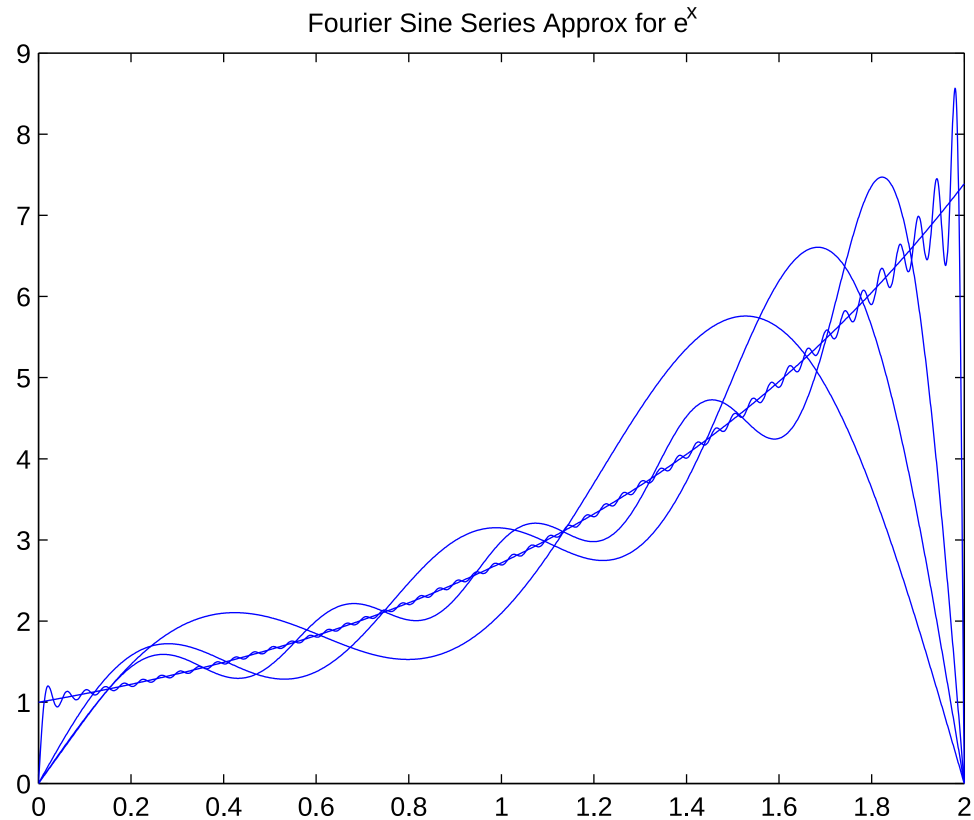
$$\begin{aligned}
A_n &= \int_0^2 e^x \sin(\frac{n\pi x}{2})\,dx \\
&= -\frac{2}{n\pi}e^x \cos(\frac{n\pi x}{2})|_0^2 + \frac{2}{n\pi}\int_0^2 e^x \cos(\frac{n\pi x}{2})\,dx \\
&= -\frac{2}{n\pi}\left(e^2(-1)^n - 1\right) + \frac{4}{n^2\pi^2}e^x \sin(\frac{n\pi x}{2})|_0^2 - \frac{4}{n^2\pi^2}\int_0^2 e^x \sin(\frac{n\pi x}{2})\,dx \\
&= -\frac{2}{n\pi}\left(e^2(-1)^n - 1\right) - \frac{4}{n^2\pi^2}\int_0^2 e^x \sin(\frac{n\pi x}{2})\,dx
\end{aligned}$$

We can add the final integral to both sides to get that

$$(1 + \frac{4}{n^2\pi^2})\int_0^2 e^x \sin(\frac{n\pi x}{2})\,dx = -\frac{2}{n\pi}\left(e^2(-1)^n - 1\right)$$

or

$$A_n = \int_0^2 e^x \sin(\frac{n\pi x}{2})\,dx = -\frac{2n\pi}{n^2\pi^2+4}\left(e^2(-1)^n - 1\right)$$



Similarily, we can find the coeffs for the cosine series by:

$$\begin{aligned} B_n &= \int_0^2 e^x \cos(\frac{n\pi x}{2})\, dx \\ &= \frac{2}{n\pi} e^x \sin(\frac{n\pi x}{2})|_0^2 - \frac{2}{n\pi} \int_0^2 e^x \sin(\frac{n\pi x}{2})\, dx \\ &= \frac{4}{n^2\pi^2} e^x \cos(\frac{n\pi x}{2})|_0^2 - \frac{4}{n^2\pi^2} \int_0^2 e^x \cos(\frac{n\pi x}{2})\, dx \\ &= \frac{4}{n^2\pi^2} \left(e^2(-1)^n - 1\right) - \frac{4}{n^2\pi^2} \int_0^2 e^x \cos(\frac{n\pi x}{2})\, dx \end{aligned}$$

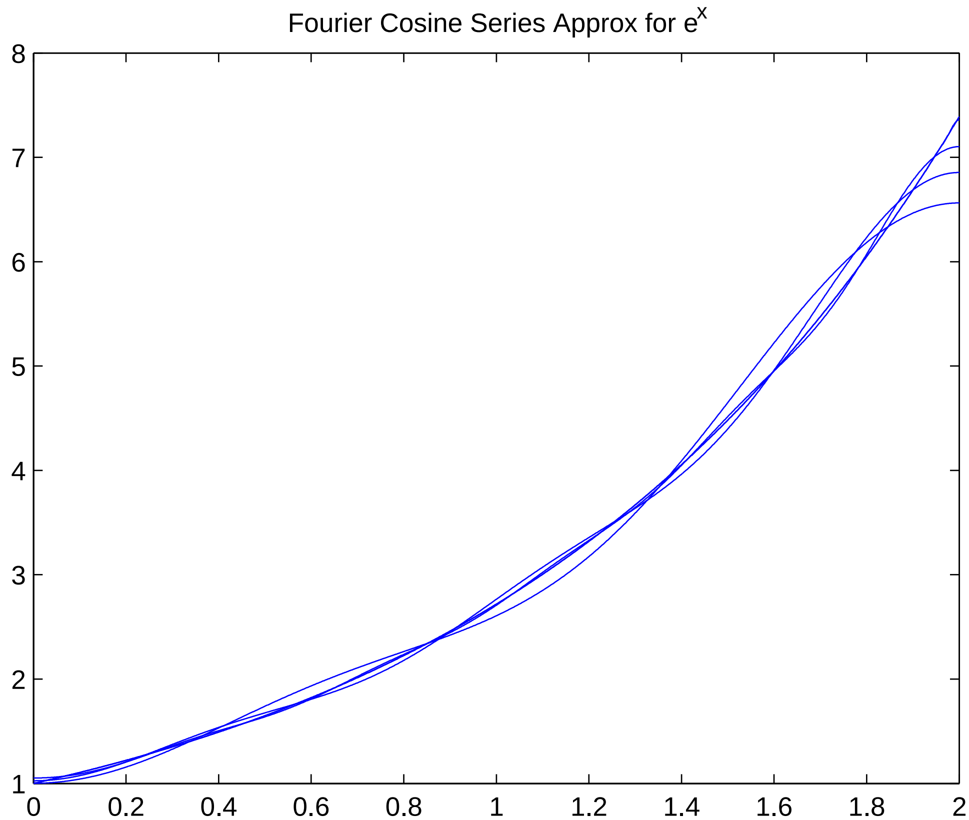
We can add the final integral to both sides to get that

$$(1 + \frac{4}{n^2\pi^2}) \int_0^2 e^x \sin(\frac{n\pi x}{2})\, dx = \frac{4}{n^2\pi^2} \left(e^2(-1)^n - 1\right)$$

or

$$B_n = \int_0^2 e^x \sin(\frac{n\pi x}{2})\, dx = \frac{4}{n^2\pi^2 + 4}(e^2(-1)^n - 1)$$

and $B_0 = \int_0^2 e^x\, dx = e^2 - 1$.



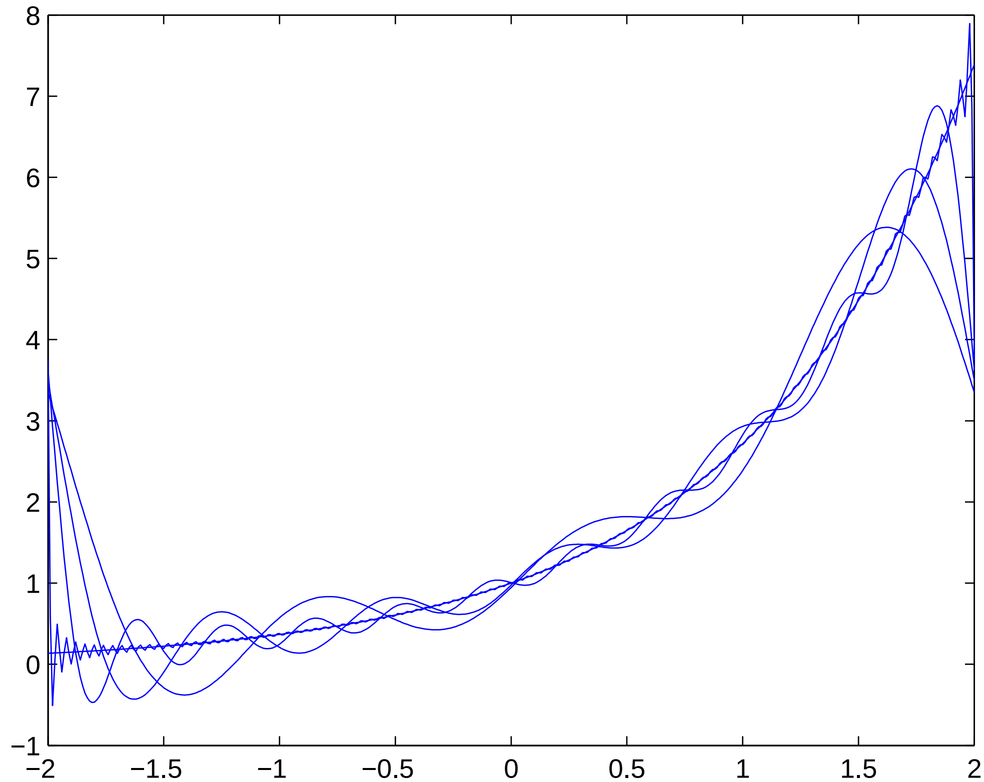
If we use the same integration by parts, but change our limits of integration to $[-2, 2]$ in order to find the full Fourier series for e^x, we get

$$A_n = \frac{1}{2}\int_{-2}^{2} e^x \sin(\frac{n\pi x}{2})\,dx$$
$$= -\frac{(-1)^n}{n\pi}(e^2 - e^{-2}) - \frac{2}{n^2\pi^2}\int_0^2 e^x \sin(\frac{n\pi x}{2})\,dx$$

or $A_n = -\frac{(-1)^n n\pi}{4+n^2\pi^2}(e^2 - e^{-2})$.
Similarly,

$$B_n = \frac{1}{2}\int_{-2}^{2} e^x \cos(\frac{n\pi x}{2})\,dx$$
$$= \frac{2(-1)^n}{n^2\pi^2}(e^2 - e^{-2}) - \frac{2}{n^2\pi^2}\int_{-2}^{2} e^x \cos(\frac{n\pi x}{2})\,dx$$

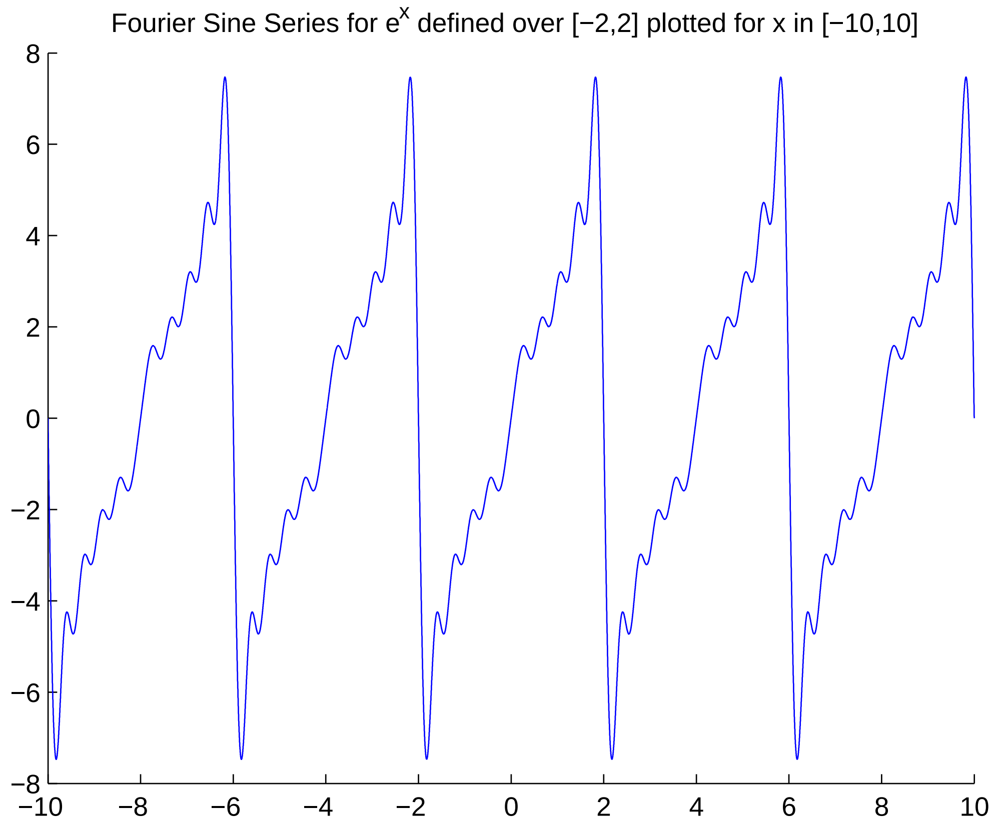
so that $B_n = \frac{2(-1)^n}{4+n^2\pi^2}(e^2 - e^{-2})$. We also have $B_0 = \frac{1}{2}\int_{-2}^{2} e^x\,dx = \frac{e^2-e^{-2}}{2}$.



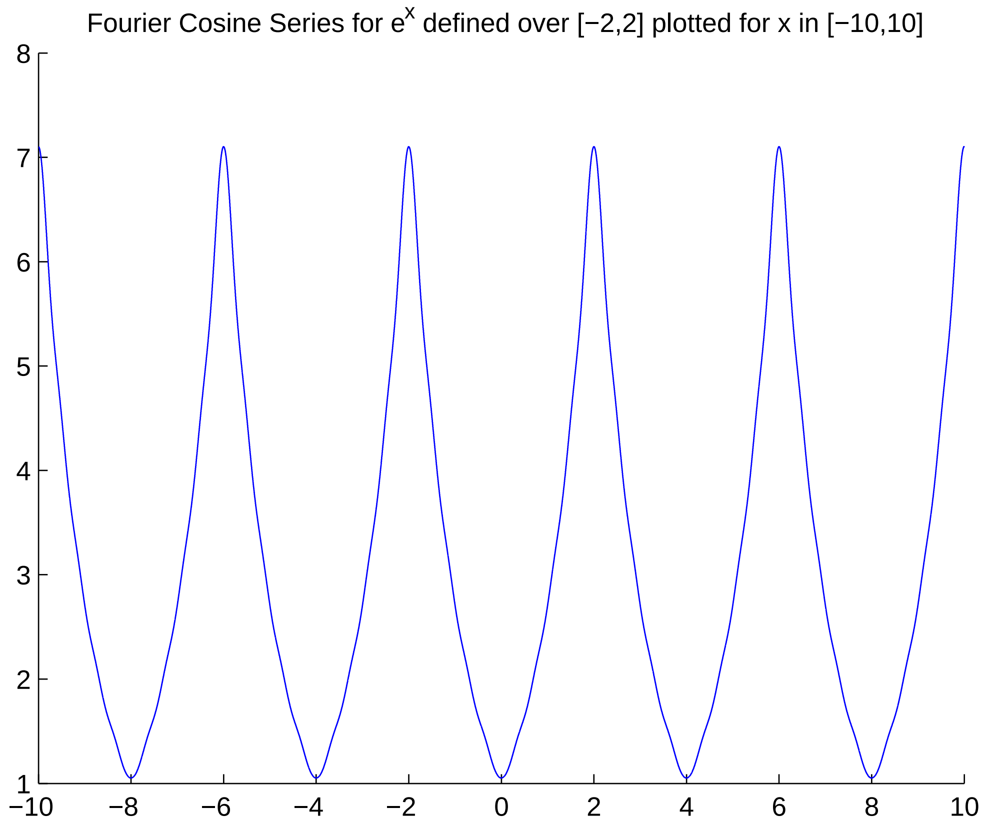
(c) Now plot (one for each FS type) just the approximate Fourier series for $x \in [-10, 10]$ with $N = 10$. What do you notice? Explain the differences in what you see.

Solution

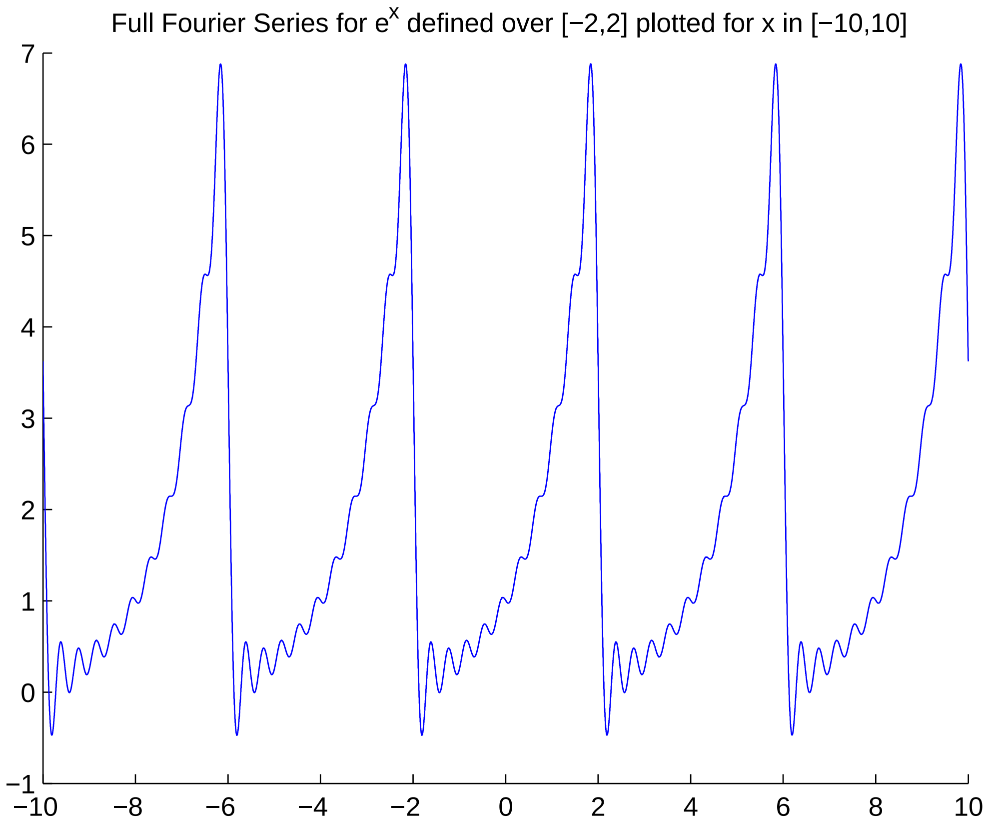
In the case of the sine series, we see the result we had on $[0, 2]$ mirrored over the line $x = y$ to $[-2, 0]$ (so that it is odd) and then the region from $[-2, 2]$ is repeated periodically over the whole line. This is because a sine series is always an odd function and of period $2l = 4$.



In the case of the cosine series, we see the result we had on $[0, 2]$ mirrored over the y-axis to $[-2, 0]$ (so that it is even) and then the region from $[-2, 2]$ is repeated periodically over the whole line. This is because a cosine series is always an even function and of period $2l = 4$.



In the case of the full series, we see the plot for the full FS obtained for $(-2, 2)$ repeated periodically of period 4 across the real line. This is because the full fourier series is always a periodic function of period $2l$, and the full fourier series was created to match e^x on the fundamental period interval $(-2, 2)$.



2. Show that IF $U(x)$ is a (steady-state) solution to $U_{xx} = 0$ on $(0, l)$ with

$$U(0) = g$$
$$U(l) = h$$

for some fixed constants g, h,and IF $\tilde{u}$ is a solution to $\tilde{u}_{xx} = \tilde{u}_t$ on $(0, l)$ with

$$\tilde{u}(0, t) = 0$$
$$\tilde{u}(l, t) = 0$$

where $\tilde{u}(x, 0) = f(x) - U(x)$, THEN $u(x, t) = \tilde{u}(x, t) + U(x)$ solves $u_{xx} = u_t$ where

$$u(0, t) = g$$
$$u(l, t) = h$$

and $u(x, 0) = f(x)$.

[**NOTE: The point of this problem is that it allows us to solve BVP's with nonhomogeneous boundary conditions by building a solution from the homogeneous b.c. problem and the corresponding steady-state problem... Notice that the seperation of variables technique breaks down if we have inhomogeneous b.c.'s]

Solution:

Letting $u = \tilde{u} + U$, we have $u_t = \tilde{u}_t + U_t = \tilde{u}_t + 0$ since U is independent of t. Also, we have $u_{xx} = \tilde{u}_{xx} + U_{xx} = \tilde{u}_{xx} + 0$, since $U_{xx} = 0$. By the PDE for $\tilde{u}$, we then have $u_t = u_{xx}$.

Finally, $u(0,t) = \tilde{u}(0,t) + U(0) = 0 + g = g$ and $u(l,t) = \tilde{u}(l,t) + U(l) = 0 + h = h$. The initial condition is obtained by $u(x,0) = \tilde{u}(x,0) + U(x) = f(x) - U(x) + U(x) = f(x)$.

3. Solve problem 8 from section 5.1 of Strauss using exercise one above.

The system we want to solve is: $u_t = u_{xx}$ on $[0,1]$, with conditions $u(0,t) = 0$, $u(1,t) = 1$, $u(x,0) = 5x/2$ for $x \in (0,2/3)$ and $u(x,0) = 3 - 2x$ for $x \in (2/3,1)$. The steady state solution to this problem satisfies $U_{xx} = 0$ and $U(0) = 0$, $U(1) = 1$, so $U(x) = x$. We need to then solve the homogeneous dirichlet bc heat equation $\tilde{u}_t = \tilde{u}_{xx}$ with $\tilde{u}(0,t) = 0$ and $\tilde{u}(1,t) = 0$, and $\tilde{u}(x,0) = f(x) - x$. We know the general solution to this BVP is $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\sin(n\pi x)$, so applying the initial condition we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = f(x) - x \ .$$

We can find the coefficients A_n by recognizing this has the form of a Fourier sine series, and so

$$A_n = 2\int_0^1 (f(x) - x)\sin(n\pi x)\,dx = 2\int_0^{2/3} \frac{3x}{2}\sin(n\pi x)\,dx + 2\int_{2/3}^1 (3 - 3x)\sin(n\pi x)\,dx \ .$$

Integrating we have

$$\begin{aligned}
A_n &= 3\left(\frac{-x}{n\pi}\cos(n\pi x)\Big|_0^{2/3} + \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^{2/3}\right) \\
&\quad + 6\left(\frac{-1}{n\pi}\cos(n\pi x)\Big|_{2/3}^1 + \frac{x}{n\pi}\cos(n\pi x)\Big|_{2/3}^1 - \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_{2/3}^1\right) \\
&= \left(\frac{-2}{n\pi}\cos(2n\pi/3) + \frac{3}{n^2\pi^2}\sin(2n\pi/3)\right) \\
&\quad + \left(\frac{-6(-1)^n}{n\pi} + \frac{6}{n\pi}\cos(2n\pi/3) + \frac{6(-1)^n}{n\pi} - \frac{4}{n\pi}\cos(2n\pi/3) - \frac{6(-1)^n}{n^2\pi^2} + \frac{6}{n^2\pi^2}\cos(2n\pi/3)\right) \\
&= \frac{6}{n^2\pi^2}\cos(2n\pi/3) - \frac{6(-1)^n}{n^2\pi^2} + \frac{3}{n^2\pi^2}\sin(2n\pi/3)
\end{aligned}$$

4. A string (with density $\rho = 1$ and tension $T = 4$) with fixed ends at $x = 0$ and $x = 10$ is hit by a hammer so that $u(x,0) = 0$ and

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} V & \text{if } x \in [-\delta + 5, \delta + 5] \\ 0 & \text{otherwise .} \end{cases}$$

Find the height of the string $u(x,t)$ for all $x \in (0,10)$ and all $t > 0$. (Your answer WILL be a bit messy...)

Solution

Our solution to the BVP is $u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin(\frac{n\pi t}{5}) + B_n \cos(\frac{n\pi t}{5})\right)\sin(\frac{n\pi x}{10}$. Applying the first initial condition we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{10} = 0$$

so that $B_n = 0$ for every n. Applying the second initial condition, we have

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{n\pi A_n}{5} \sin(\frac{n\pi x}{10} \begin{cases} V & \text{if } x \in [-\delta+5, \delta+5] \\ 0 & \text{otherwise .} \end{cases}$$

We can now use the fact that we have a Fourier sine series here to find the coefficients A_n. We will need

$$\frac{n\pi A_n}{5} = \frac{2}{10} \int_{5-\delta}^{5+\delta} V \sin(\frac{n\pi x}{10})\, dx = \frac{-2V}{n\pi} \cos(\frac{n\pi x}{10})|_{5-\delta}^{5+\delta}$$

Thus,

$$A_n = \frac{-10V}{n^2\pi^2} \left(\cos(\frac{n\pi}{2} + \frac{n\pi\delta}{10}) - \cos(\frac{n\pi}{2} - \frac{n\pi\delta}{10}) \right)$$

5. Problem 15 section 5.2 of Strauss.

   Solution

   Since $|\sin(x)|$ is an even function, the coefficients for the *sine* terms in the full fourier series will vanish (=0) and we will have a pure cosine series.

6. Using parts of our discussion in class, solve the fourth order equation $u_{xxxx} = u_t$ if $u(0,t) = 0$, $u(3,t) = 0$, $u_{xx}(0,t) = 0$, and $u_{xx}(3,t) = 0$.

   Solution

   Using separation of variables, we have $\frac{X^{(4)}}{X} = \frac{T'}{T} = \lambda$. We know by our work in class that since we have the above homogeneous boundary conditions, we will have eigenvalues $\lambda \geq 0$ only. So, we will check the cases $\lambda = 0$ and $\lambda > 0$. If $\lambda = 0$, we have $X(x) = ax^3 + bx^2 + cx + d$ and applying our boundary conditions we get: $d = 0$, $27a + 9b + 3c = 0$, $2b = 0$ and $18a = 0$. Thus $a = b = c = d = 0$ and we get only the trivial solution.

   Now we check $\lambda = \beta^4 = 0$. Then the characteristic equation for $X^{(4)} - \beta^4 X = 0$ is $r^4 - \beta^4 =$. We can factor this as $(r^2 - \beta^2)(r^2 + \beta^2) = 0$ so that the roots are $r = \pm\beta$, $\pm i\beta$ and $X(x) = c_1 e^{\beta x} + c_2 e^{-\beta x} + c_3 \cos(\beta x) + c_4 \sin(\beta x)$. Applying the first three boundary conditions we get:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 e^{3\beta} + c_2 e^{-3\beta} + c_3 \cos(3\beta) + c_4 \sin(3\beta) &= 0 \\ \beta^2(c_1 + c_2 - c_3) &= 0 \end{aligned}$$

   This tells us that $c_3 = 0$ and so $c_2 = -c_1$. Subbing that into the second equation gives $c_1(e^{3\beta} - e^{-3\beta}) + c_4 \sin(3\beta) = 0$. Now applying our final condition gives $\beta^2(c_1 e^{3\beta} - c_1 e^{-3\beta} - c_4 \sin(3\beta)) = 0$. This leads us to conclude that $c_1 = 0$ and so $c_4 \sin(3\beta) = 0$. Our only hope for a nontrivial solution is to have $3\beta = n\pi$ or $\beta = \frac{n\pi}{3}$. Thus we get infinitely many solutions for X, one for each n, and

$$X_n(x) = C_n \sin(\frac{n\pi x}{3}) \ .$$

   Now we can solve for the corresponding functions T_n with $\lambda_n = \beta^4 = (\frac{n\pi}{3})^4$. $T_n' = \frac{n^4\pi^4}{81} T_n$, so that $T_n = D_n e^{\frac{n^4\pi^4 t}{81}}$. Our final solution is then

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{\frac{n^4\pi^4 t}{81}} \sin(\frac{n\pi x}{3}) \ .$$

Aside: Because our BC's meet the symmetry condition for operator $L(u) = u_{xxxx}$ which we discussed in class, we know that if we had an initial condition, at this point we can use the Fourier series method to uniquely determine the coefficients A_n.