Homework Set # 5 – SOLUTIONS – Math 435

1. Consider the Fourier **sine** series of each of the following functions. Do not compute the coefficients, but use the pointwise convergence theorem to discuss the convergence of each of the series. Explain what you would expect to see if you plotted a partial sum approximation via F.S. of each function, paying attention to what would happen at the endpoints.

   (a) $f(x) = x^3$ on $(0, l)$

   (b) $f(x) = lx - x^2$ on $(0, l)$

   (c) $f(x) = \frac{1}{x^2}$ on $(0, l)$.

   Solutions:

   (a) Since $f(x) = x^3$ and $f'(x) = 3x^2$ are continuous functions on [0, l] for any $l \in \mathbb{R}$, we know by the pointwise convergence theorem that the fourier series for f will converge to f on $(0, l)$. There is a mismatch between the value of x^3 and that of the sine functions $sin(n\pi x/l)$ at $x = l$, so I would expect the partial sums to approach l^3 for x near l, oscillating in trying to do so, and then abruptly diving down to 0 at $x = l$. As the number of terms in the partial sum increases, though, the region over which we see this behavior shrinks away.

   (b) Again, f and f' are continuous on $[0, l]$, so the fourier series converges to f on $(0, l)$ pointwise. This time, there is no mismatch between the boundary values of f and the sine functions so the partial sums will converge pointwise on $[0, l]$. In fact, the uniform convergence theorem applied here, and we will get uniform convergence of the Fourier series to f on $[0, l]$.

   (c) Since $f(x)$ for this case is NOT continuous on $[0, l]$ (undefined at $x = 0$), we are not guaranteed pointwise convergence of the fourier series to $f(x)$ on $(0, l)$. The numerical approximation for the partial sums may be misleading in this case, so its important to take care when looking at computer approximations (careful what you believe) when you are not guaranteed convergence.

2. Use the pointwise convergence theorem to explain why it may be that the Fourier series of a given function could converge even though the Fourier series of it's derivative might not.

   Solution:

   The pointwise convergence theorem requires that we know that the derivative of a function be piecewise continuous also in order to say anything about convergence of the F.S. for the function itself. Thus, we could have a function $f(x)$ which is piecewise continuous and has a piecewise continuous derivative, so that its F.S. converges pointwise over $(0, l)$ to $f(x)$. Now, we know that $f'(x)$ is piecewise continuous, but it is possible that $f''(x)$ is NOT, thus giving no guarantees about convergence of the F.S. for $f'(x)$.

3. Let

$$\phi(x) = \begin{cases} 1 - (1-x)^2 & \text{for } 0 \le x < 1 \\ (1-x)^2 & \text{for } 1 \le x \le 2 \end{cases}$$

   (a) Find the fourier sine series for $\phi(x)$ over $(0, 2)$.

   (b) Does the series converge pointwise to $\phi(x)$?

(c) Explain what the Gibbs phenomenon tells you about the behavior of the partial sums of the Fourier Series for ϕ near $x = 1$. [Note: you might find it useful to look at partial sum approximations via a computer so that you can see (b) and (c) "in action"]

Solution:

(a) We want to express $\phi(x)$ in terms of a F. sine S. So, we want

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/2) \ .$$

We have

$$\begin{aligned}
a_n &= \int_0^2 \phi(x) \sin(n\pi x/2)\, dx \\
&= \int_0^1 (1-(1-x)^2) \sin(n\pi x/2)\, dx + \int_1^2 (1-x)^2 \ \sin(n\pi x/2)\, dx \\
&= -\frac{2\cos(n\pi x/2)}{n\pi}|_0^1 - \left(-\frac{2(1-x)^2\cos(n\pi x/2)}{n\pi}|_0^1 + \int_0^1 \frac{4(x-1)\cos(n\pi x/2)}{n\pi}\, dx\right) \\
&+ \left(-\frac{2(1-x)^2\cos(n\pi x/2)}{n\pi}|_1^2 + \int_1^2 \frac{4(x-1)\cos(n\pi x/2)}{n\pi}\, dx\right) \\
&= \frac{-2\cos(n\pi/2)+2}{n\pi} - \frac{2}{n\pi} - \int_0^1 \frac{4(x-1)\cos(n\pi x/2)}{n\pi}\, dx - \frac{2(-1)^n}{n\pi} + \int_1^2 \frac{4(x-1)\cos(n\pi x/2)}{n\pi}\, dx \\
&= \frac{-2\cos(n\pi/2)-2(-1)^n}{n\pi} + \int_0^1 \frac{8\sin(n\pi x/2)}{n^2\pi^2}\, dx - \int_1^2 \frac{8\sin(n\pi x/2)}{n^2\pi^2} \\
&= \frac{-2\cos(n\pi/2)-2(-1)^n}{n\pi} - \frac{16\cos(n\pi/2)-16}{n^3\pi^3} + \frac{16(-1)^n-16\cos(n\pi/2)}{n^3\pi^3} \ .
\end{aligned}$$

(b) By the pointwise convergence theorem, we get pointwise convergence to $\phi(x)$ for the F.S. everywhere in $(0, 2)$ except at the jump at $x = 1$, since ϕ is clearly piecewise continuous and since

$$\phi'(x) = \begin{cases} 2-2x & \text{on } 0 < x < 1 \\ -2+2x & \text{for } 1 < x < 2 \end{cases}$$

we know $\phi'(x)$ is also piecewise continuous on $(0, 2)$. At $x = 1$, the F.S. converges to $1/2$, while $\phi(1) = 0$.

(c) The Gibbs phenomenon tells us that *near* the jump, we can expect that any partial sum of the Fourier Series will have an overshoot of about 9% of the jump size. Since our jump size is 1, we can expect overshoots on either side of the jump of about 0.09. This is true for ANY partial sum of the Fourier series, no matter how many terms are included, but the more terms that are included, the closer to the jump the overshoot will lie.

4. Following the steps below solve the fourth order BVP $u_{xxxx} = u_t$ if $u(0,t) = 0$, $u(3,t) = 0$, $u_{xx}(0,t) = 0$, and $u_{xx}(3,t) = 0$, and explain how to satisfy an initial condition $u(x, 0) = \phi(x)$.

(a) Using the method we discussed in class Friday, show that all the eigenvalues of the operator $L(X) = X''''$ are positive.

(b) Use seperation of variables and the information you obtained in (a) to find the general solution to the BVP.

(c) Show using the method discussed in class on Thursday that all the eigenfunctions of L given the boundary conditions above are orthogonal to one another, and explain how to apply an intial condition to the solution of the BVP you found in (b).

(a) Let λ be an eigenvalue of L and X be an eigenfunction so that $L(X) = \lambda X$.

$$\begin{aligned}\lambda \int_0^3 X^2\,dx &= \int_0^3 X''''X\,dx \\ &= X'''X|_0^3 - \int_0^3 X'''X'\,dx \\ &= -\left(X''X'|_0^3 - \int_0^3 (X'')^2\,dx\right) \\ &= \int_0^3 (X'')^2\,dx\end{aligned}$$

so that all eigenvalues λ for this BVP must be positive.

(b) Try solutions u of the form $u(x,t) = X(x)T(t)$. Then for some constant λ we have $X'''' = \lambda X$ and $T' = \lambda T$. We also have that for nontrivial solutions, $X(0) = 0 = X(3)$ and $X''(0) = 0 = X''(3)$. Solving for X, we know that we should try $X(x) = e^{rx}$ and that we will only have nontrivial solutions for positive λ by part (a). Let $\lambda = \beta^4$, then we know $r^4 = \beta^4$ and $r = \pm\beta, \pm i\beta$, so $X(x) = c_1e^{\beta x} + c_2e^{-\beta x} + c_3\cos(\beta x) + c_4\sin(\beta x)$ are the real valued solutions to ODE for X. Now if we apply the boundary conditions, we see that in fact $X_n(x) = a_n\sin(\frac{n\pi x}{3})$ is a solution to the BVP for each $n \in \mathbb{Z}$, where $\beta = \frac{n\pi}{3}$, so that these eigenfunctions each correspond to eigenvalues $\lambda_n = (\frac{n\pi}{3})^4$. Finally if we find T_n corresponding to X_n for each n, we have $T_n = C_ne^{(n^4\pi^4t)/(3^4)}$, and $u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$.

(c) Suppose now that $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of L. Then

$$\begin{aligned}(\lambda_1 - \lambda_2)\int_0^3 X_1X_2\,dx &= \int_0^3 X''''X_2 - X_1X_2'''' \\ &= \left(X_1'''X_2 - X_1X_2'''\right)|_0^3 - \int_0^3 X_1'''X_2' - X_1'X_2'''\,dx \\ &= -\left(X_1''X_2' - X_1'X_2''\right)|_0^3 + \int_0^3 (X_1''X_2'' - X_1''X_2'')\,dx \\ &= 0\end{aligned}$$

Thus, eigenfunctions of L corresponding to distinct eigenvalues of L are orthogonal. Applying an initial condition gives $u(x,0) = \sum_{n=1}^{\infty} X_n(x) = \sum_{n=1}^{\infty}(a_n\sin(\frac{n\pi x}{3})) = \phi(x)$. Since all the eigenfunctions are orthogonal on $(0,3)$, we have for all $n = 1, 2, 3, \ldots$

$$a_n = \frac{\int_0^3 \phi(x)\sin(\frac{n\pi x}{3})\,dx}{\int_0^3 \sin^2(\frac{n\pi x}{3})\,dx}\ .$$

So long as ϕ is at least piecewise continuous and has a p.w. cts first derivative, we know that the resulting fourier series really will converge to ϕ on $(0,3)$, and this gives us a valid way to determine the coefficients a_n in u to obtain a unique solution to the IBVP.

5. Solve the equation $u_t = u_{xx}$ in $(0, 2)$ with $u_x(0, t) = 1$ and $u_x(2, t) = t$ for all $t > 0$, assuming that $u(x, 0) = 0$. Hint: write u and it's derivatives in terms of Fourier cosine series and proceed as in class on Friday. **Solution:**

Let $u(x,t) = \frac{1}{2}u_0(t) + \sum_{n=1}^{\infty} u_n(t)\cos(n\pi x/2)$. Then

$$u_0(t) = \int_0^2 u(x,t)\,dx$$

and

$$u_n(t) = \int_0^2 u(x,t)\cos(n\pi x/2)\,dx$$

for $n > 0$. Also, we need $u_t(x,t) = \frac{1}{2}v_0(t) + \sum_{n=1}^{\infty} v_n(t)\cos(n\pi x/2)$, so that

$$v_0(t) = \int_0^2 u_t(x,t)\,dx = \frac{du_0}{dt}$$

and

$$v_n(t) = \int_0^2 u_t(x,t)\cos(n\pi x/2)\,dx = \frac{du_n}{dt} \ .$$

Finally, let $u_{xx}(x,t) = \frac{1}{2}w_0(t) + \sum_{n=1}^{\infty} w_n(t)\cos(n\pi x/2)$, so that

$$w_0(t) = \int_0^2 u_{xx}(x,t)\,dx = u_x(x,t)|_0^2 = t - 1$$

and

$$w_n(t) = \int_0^2 u_{xx}(x,t)\cos(n\pi x/2)\,dx = u_x(x,t)\cos(n\pi x/2)|_0^2 + \frac{n\pi}{2}\int_0^2 u_x \sin(n\pi x/2)\,dx \ .$$

Subbing in and integrating by parts once more, we get

$$w_n(t) = [t(-1)^n - 1] - \frac{n^2\pi^2}{4}u_n(t) \ .$$

Let $\lambda_n = n^2\pi^2/4$ in what follows. Now we can use the PDE to link the v_i's and w_i's to get

$$\frac{du_0}{dt} = t - 1$$

and

$$\frac{du_n}{dt} = [(-1)^n t - 1] - \lambda_n u_n \ .$$

We can solve easity for u_0 by integrating to get $u_0(t) = .5t^2 - t + c_0$. To find the other u_n's we need to solve the ODE

$$u_n' + \lambda_n u_n = (-1)^n t - 1 \ .$$

We can multiply both sides by the integrating factor $e^{\lambda_n t}$ to get

$$(u_n e^{\lambda_n t})' = e^{\lambda_n t}((-1)^n t - 1)$$

so that

$$e^{\lambda_n t}u_n = (-1)^n[\frac{te^{\lambda_n t}}{\lambda_n} - \frac{e^{\lambda_n t}}{\lambda_n^2}] - \frac{e^{\lambda_n t}}{\lambda_n} + c_n \ .$$

Finally,

$$u_n(t) = \frac{(-1)^n}{\lambda_n^2}[\lambda_n t - 1] - \frac{1}{\lambda_n} + c_n e^{-\lambda_n t} \ ,$$

so that

$$u(x,t) = \frac{t^2}{4} - \frac{t}{2} + \frac{c_0}{2} + \sum_n u_n(t) \cos(n\pi x/2) \ .$$

Now we need to satisfy the initial condition $u(x,0) = 0$. Plugging in to our solution, we get

$$u(x,0) = \frac{c_0}{2} + \sum_n \left(\frac{(-1)^{n+1}}{\lambda_n^2} - \frac{1}{\lambda_n} + c_n \right) \cos(n\pi x/2) = 0$$

so that we can just choose $c_0 = 0$ and $c_n = \frac{(-1)^n}{\lambda_n^2} + \frac{1}{\lambda_n}$ for all $n > 0$.