Homework Set # 4 – Math 435 Summer SOLUTIONS

1. Solve the heat equation (i.e. - the diffusion equation) $4u_{xx} = u_t$ on a rod of length 2 if $u(x,0) = sin(\frac{\pi x}{2})$ and $u(0,t) = 0 = u(2,t)$.

   Solution:

   We are solving the heat equation on a finite interval $(0,2)$, with dirichlet boundary conditions, so we can use the general solution to this boundary value problem that we derived in class via seperation of variables:

   $$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin(n\pi x/2) \ .$$

   In order to finish, we need to determine the values of the A_n's. Since

   $$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/2) = \sin(\pi x/2)$$

   we can take $A_1 = 1$ and $A_n = 0$ for all $n \neq 1$. Thus

   $$u(x,t) = e^{-\pi^2 t} \sin(\pi x/2) \ .$$

2. Solve the wave equation $3u_{xx} = u_{tt}$ for a clamped string of length $l = 1$ (so $u(0,t) = 0 = u(1,t)$) such that $u(x,0) = 2\sin(\pi x)\cos(\pi x)$ and $u_t(x,0) = 0$. [hint: use a double angle identity from trig]

   Again, we are solving the wave equation on a finite length interval $(0,1)$ with dirichlet boundary conditions, so since we have already solved this general boundary value problem, we can use the solution we obtained via seperation of variables:

   $$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(n\pi\sqrt{3}t) + b_n \sin(n\pi\sqrt{3}t)\right) \sin(n\pi x) \ .$$

   We need to determine the values of the a_n's and b_n's in order to have solved our problem completely. Since

   $$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = 2\sin(\pi x)\cos(\pi x) = \sin(2\pi x)$$

   we see that we can take $a_2 = 1$ and $a_n = 0$ for $n \neq 2$. This gives

   $$u(x,t) = \cos(2\pi\sqrt{3}t)\sin(2\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi\sqrt{3}t)\sin(n\pi x) \ .$$

   It follows that

   $$u_t(x,t) = -2\pi\sqrt{3}\sin(2\pi\sqrt{3}t)\sin(2\pi x) + \sum_{n=1}^{\infty} n\pi\sqrt{3}b_n \cos(n\pi\sqrt{3}t)\sin(n\pi x) \ ,$$

   and

   $$u_t(x,0) = \sum_{n=1}^{\infty} n\pi\sqrt{3}b_n \sin(n\pi x) = 0 \ .$$

   This tells us that we can take $b_n = 0$ for all n. Finally, we have

   $$u(x,t) = \cos(2\pi\sqrt{3}t)\sin(2\pi x) \ .$$

3. Strauss Exercise 4, pg 87 (solve by seperation of variables, in the same way that we did in class)

Letting $u(x,t) = X(x)T(t)$ and subbing in, we have

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = \lambda$$

so that

$$X'' = \lambda X$$
$$T'' + rT' = \lambda c^2 T \ .$$

We again have Dirichlet boundary conditions, so the solution for X is only nontrivial if $\lambda = -\beta^2 < 0$, and we get $X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$. Applying the boundary conditions, we have $X(0) = c_1 = 0$ and $X(l) = c_2\sin(\beta l) = 0$. In order to get a nontrivial solution, we then need that $\beta l = n\pi$ for some $n \in \mathbb{Z}$, or $\beta = n\pi/l$. This tells us that we have a solution X_n for each integer n, and

$$X_n(x) = c_n \sin(n\pi x/l) \ .$$

Now we can solve for the corresponding functions T_n. We can try $T_n(t) = e^{kt}$. This gives

$$k^2 + rk - c^2\lambda = 0$$

which has as it's solutions

$$k = \frac{-r \pm \sqrt{r^2 + 4c^2\lambda}}{2} = \frac{-r \pm \sqrt{r^2 - 4c^2n^2\pi^2/l^2}}{2} \ .$$

The types of solutions we get then depend on whether or not k is real or complex, which is determined by the sign of $r^2 - 4c^2n^2\pi^2/l^2$. Since we are given that $0 < r < 2\pi c/l$ that implies that $r^2 < 4\pi^2c^2/l^2$ and since $n \geq 1$, we get $r^2 < 4\pi^2c^2n^2/l^2$, or $r^2 - 4c^2n^2\pi^2/l^2 < 0$. Hence k is complex and the solutions are

$$T_n(t) = e^{-rt/2}\left(a_n\cos(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2}t) + b_n\sin(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2}t)\right) \ .$$

This gives us, by the linearity of the PDE and the superposition principle, thatthe general solution is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-rt/2}\left(a_n\cos(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2}t) + b_n\sin(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2}t)\right)\sin(n\pi x/l) \ .$$

Now to determine the coefficients, we note that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/l) = \phi(x) \ ,$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(-\frac{r}{2}a_n + \frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2}b_n\right)\sin(n\pi x/l) = \psi(x) \ .$$

so that the a_n's and b_n's can be found by the method Fourier sine coefficients.

[NOTE: the neat thing about this problem is that you can directly see that this really does give you a damped wave - because of the factor $e^{-rt/2}$ multiplying onto every term, as $t \to \infty$, $u(x,t) \to 0$. This is quite different from the solution to the nondamped wave equation, where waves perpetuate indefinitely, with no decrease in amplitude.]

4. Straus, Exercise 6, pg 89.

   We let $u(x,t) = X(x)T(t)$ and substitute in the PDE $tu_t = u_{xx} + 2u$. This yields $tXT' = X''T + 2XT$, which can be separated into the two ODE's:

$$\frac{tT'}{T} - 2 = \lambda$$

   and

$$\frac{X''}{X} = \lambda \ .$$

   Since we have homogeneous Dirichlet Boundary conditions, and we are working with our usual ODE for $X(x)$, we know that $X_n(x) = C_n sin(nx)$ for each n in the integers are all the possible solutions. Now solving the ODE for T, we have

$$tT' - (2+\lambda)T = 0,$$

   or

$$T' - \frac{2+\lambda}{t}T = 0 \ .$$

   Separating variables, we get:

$$\frac{T'}{T} = (\lambda+2)/t$$

   so integrating both sides yields

$$\ln|T| = (\lambda+2)ln|t| + C$$

   or

$$T = Ce^{(\lambda+2)ln|t|} = Ct^{\lambda+2}$$

   Since there is a value of λ for each integer n by

$$\lambda = -n^2$$

   , we have

$$T_n(t) = D_n t^{-n^2+2} \ .$$

   Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n t^{(-n^2+2)} sin(nx)$$

   is the general solution to our BVP.

   Now, applying the inital condition we see

$$u(x,0) = 0$$

   regardless of our choices of the values for A_n's! So ANY values of A_n's work and we get infinitely many possible solutions to the IBVP. This problem is not well-posed.

5. Strauss, exercise 1, page 92

   We have $kX''T = XT'$, so that

$$\frac{X''}{X} = \frac{T'}{kT} = \lambda$$

. We know that the general solution for X is $X = c_1 e^{rt} + c_2 e^{-rt}$, where $r = \pm\sqrt{\lambda}$. We can first check the case where $\lambda > 0$, or $\lambda = \beta^2$. This yields $X(x) = c_1 e^{\beta x} + c_2 e^{-\beta x}$ and we can apply the boundary conditions $X(0) = 0$ and $X'(l) = 0$. We then have $c_1 = -c_2$ and $c_1 \beta(e^{\beta l} + e^{-\beta l}) = 0$. In order for the latter to be true, we need $c_1 = 0$ and so we have only the trivial solution $X(x) = 0$ for all x.

Now we can check the case for $\lambda = 0$. This gives $X(x) = cx + d$, and applying the boundary conditions we have $d = 0$ and $c = 0$, so that again we get only the trivial solution.

Finally we look at the case $\lambda < 0$ or $\lambda = -\beta^2$. This yields $X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$. Applying $X(0) = 0$ gives $c_1 = 0$. Applying next that $X'(l) = 0$ gives $c_2 \beta \cos(\beta l) = 0$ so that in order to obtain something nontrivial, we must take $\beta l = \frac{(2n-1)\pi}{2}$, or $\beta = \frac{(2n-1)\pi}{2l}$. We then see we have an infinite family of solutions $X_n(x) = c_n \sin\left(\frac{(2n-1)\pi x}{2l}\right)$.

We can proceed to find the solutions $T_n(t)$ associated to each $X_n(x)$. For a fixed n, $\lambda = -\frac{(2n-1)^2\pi^2}{4l^2}$, so the equation for T_n is

$$T_n' = -\frac{k(2n-1)^2\pi^2}{4l^2} T_n$$

and the solution is $T_n(t) = D_n e^{-\frac{k(2n-1)^2\pi^2 t}{4l^2}}$. So, for each n, we have a solution $u_n(x,t) = A_n e^{-\frac{k(2n-1)^2\pi^2 t}{4l^2}} \sin\left(\frac{(2n-1)\pi x}{2l}\right)$ to the boundary value problem, and the general solution is then (by the superposition principle and the fact that our PDE is linear)

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{k(2n-1)^2\pi^2 t}{4l^2}} \sin\left(\frac{(2n-1)\pi x}{2l}\right) .$$

We are given no initial condition for this problem, so we are done!

6. Show that IF $U(x)$ is a (steady-state) solution to $U_{xx} = 0$ on $(0, l)$ with

$$U(0) = g$$
$$U(l) = h$$

for some fixed constants g, h,and IF $\tilde{u}$ is a solution to $\tilde{u}_{xx} = \tilde{u}_t$ on $(0, l)$ with

$$\tilde{u}(0,t) = 0$$
$$\tilde{u}(l,t) = 0$$

where $\tilde{u}(x, 0) = f(x) - U(x)$, THEN $u(x,t) = \tilde{u}(x,t) + U(x)$ solves $u_{xx} = u_t$ where

$$u(0,t) = g$$
$$u(l,t) = h$$

and $u(x, 0) = f(x)$.

[**NOTE: The point of this problem is that it allows us to solve BVP's with nonhomogeneous boundary conditions by building a solution from the homogeneous b.c. problem and the corresponding steady-state problem... Notice that the seperation of variables technique breaks down if we have inhomogeneous b.c.'s]

7. Solve problem 8 from section 5.1 of Strauss using exercise 6 above.

Solving the steady state system for U, we get

$$\int U_{xx}\,dx = \int 0\,dx$$

implies

$$U_x = C_1$$

and then integrating again, we get

$$U(x) = C_1 x + C_2\ .$$

Applying the boundary conditions $U(0) = 0$ and $U(1) = 1$ results in $U(x) = x$.

Now, we need to find the solution to the corresponding homogeneous problem $\tilde{u}$. Since it satisfies $\tilde{u}_t = \tilde{u}_{xx}$ on $(0, 1)$ with $\tilde{u} = \phi(x) - x$ and $\tilde{u}(0, t) = 0 = \tilde{u}(1, t)$, we know that the solution can be found by seperation of variables. Since we have Dirichlet boundary conditions and it's the heat equation, we get

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-n^2\pi^2 t}\ .$$

Now solving for the A_n's can be done by the standard means of finding Fourier sine series coefficients, since

$$\tilde{u}(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \phi(x) - x\ .$$

This means that

$$A_n = \int_0^1 (\phi(x) - x)) \sin(n\pi x)\,dx$$

Since

$$\phi(x) = \begin{cases} \frac{5x}{2} & \text{for } 0 < x < 2/3 \\ 3 - 2x & \text{for } 2/3 < x < 1 \end{cases}$$

we get

$$\phi(x) - x = \begin{cases} \frac{3x}{2} & \text{for } 0 < x < 2/3 \\ 3 - 3x & \text{for } 2/3 < x < 1 \end{cases}$$

so

$$A_n = \int_0^{2/3} \frac{3x}{2} \sin(n\pi x)\,dx + \int_{2/3}^1 (3 - 3x) \sin(n\pi x)\,dx\ .$$

Now if we use integration by parts, we can find the generic integral

$$\begin{aligned} \int_a^b x \sin(n\pi x)\,dx &= -\frac{x}{n\pi} \cos(n\pi x)\Big|_a^b + \int_a^b \frac{1}{n\pi} \cos(n\pi x\,dx \\ &= \frac{1}{n\pi}[-b\cos(n\pi b) + a\cos(n\pi a)] + \frac{1}{n^2\pi^2}[\sin(n\pi b) - \sin(n\pi a)] \end{aligned}$$

which we can use to get A_n. Subbing in we get

$$A_n = \frac{3}{2}\left(-\frac{2}{3n\pi}\cos(2n\pi/3) + \frac{1}{n^2\pi^2}\sin(2\pi n/3)\right) - \frac{3}{n\pi}[\cos(n\pi) - \cos(2n\pi/3)]$$
$$- 3\left(\frac{1}{n\pi}(-\cos(n\pi) + \frac{2}{3}\cos(2n\pi/3)) - \frac{1}{n^2\pi^2}\sin(2n\pi/3)\right) .$$

This simplifies to

$$A_n = \frac{9}{2n^2\pi^2}\sin(2n\pi/3) .$$

Plugging these coefficients into the expansion for $\tilde{u}$ defines $\tilde{u}$ completely. Finally we get $u(x,t) = \tilde{u} + x$ to be the solution to our inhomogeneous problem.

8. A string (with density $\rho = 1$ and tension $T = 4$) with fixed ends at $x = 0$ and $x = 10$ is hit by a hammer so that $u(x, 0) = 0$ and

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} V & \text{if } x \in [-\delta + 5, \delta + 5] \\ 0 & \text{otherwise .} \end{cases}$$

Find the height of the string $u(x,t)$ for all $x \in (0, 10)$ and all $t > 0$. (Your answer WILL be a bit messy...)

Solution:

Again, we have the wave equation with dirichlet boundary conditions, so that the solution looks like

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \sin(2n\pi t/10) + B_n \cos(2n\pi t/10)) \sin(n\pi x/10) .$$

In order to satisfty our initial conditions, we note that

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/10) = 0 ,$$

which tells us that $B_n = 0$ for all n, and

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi t/5)\sin(n\pi x/10) .$$

Now for the inital velocity:

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi A_n}{5}\cos(n\pi t/5)\sin(n\pi x/10) ,$$

so

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi A_n}{5}\sin(n\pi x/10) .$$

Thus we are expanding our initial velocity in a fourier sine series, so the coefficients are

$$\frac{n\pi}{5}A_n = \frac{1}{5}\int_0^{10} u_t(x,0)\sin(n\pi x/10)\,dx$$

and using the definition of $u_t(x,0)$ we get

$$\frac{n\pi}{5}A_n = \frac{1}{5}\int_{5-\delta}^{5+\delta} V\sin(n\pi x/10)\,dx \ .$$

Doing the computation gives

$$A_n = \frac{10V}{n^2\pi^2}[\cos((5-\delta)n\pi/10) - \cos((5+\delta)n\pi/10)] \ .$$

To this point is fine, but we could also simplify further using the fact that

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \ .$$

This yields

$$A_n = \frac{20V}{n^2\pi^2}\sin(5n\pi/10)\sin(\delta n\pi/10)$$

which we can sub into

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi t/5)\sin(n\pi x/10)$$

to get our final solution u.

9. Problems 5a and 6a from section 5.1 of Strauss, relying on the FS (sine) we found for $f(x) = x$ on (0,l) in class (and in the book).

(5a) Since $x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2l}{n\pi}\sin(n\pi x/l)$, we can integrate the series term-by-term to get

$$\frac{x^2}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2l^2}{n^2\pi^2}\cos(n\pi x/l) + C \ .$$

So, it just remains to determine C. Note that this gives us a Fourier cosine sereis for $\frac{x^2}{2}$, so the C should be the same as the $\frac{1}{2}A_0$ of the cosine series. Thus, since

$$A_0 = \frac{2}{l}2\int_0^l \frac{x^2}{2}\,dx = l^2 3$$

we get $C = l^2 6$, and

$$\frac{x^2}{2} = \frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n 2l^2}{n^2\pi^2}\cos(n\pi x/l) \ .$$

(6a) Now, we can do basically the same thing to get a Fourier series expansion for x^3. Integrating the series for $\frac{x^2}{2}$ term-by-term gives

$$\frac{x^3}{6} = \frac{l^2}{6}x + \sum_{n=1}^{\infty} \frac{(-1)^n 2l^3}{n^3\pi^3}\sin(n\pi x/l) + C \ .$$

Thus

$$x^3 = l^2 x + \sum_{n=1}^{\infty} \frac{(-1)^n 12l^3}{n^3\pi^3}\sin(n\pi x/l) + C \ .$$

Again, we still need to determine C, but subbing in zero to both sides shows us that $C = 0$. We aren't quite done because the $l^2 x$ term makes the right hand side not quite a Fourier series. If we sub in the sine series for x, we get

$$x^3 = l^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2l}{n\pi} \sin(n\pi x/l) + \sum_{n=1}^{\infty} \frac{(-1)^n 12 l^3}{n^3 \pi^3} \sin(n\pi x/l) \ ,$$

or

$$x^3 = \sum_{n=1}^{\infty} (-1)^n \left(\frac{-2l^3}{n\pi} + \frac{12 l^3}{n^3 \pi^3} \right) \sin(n\pi x/l) \ .$$

10. Problem 15 section 5.2 of Strauss.

Since $|\sin(x)|$ is an even function, the sine coeffients for the full Fourier series over $(-\pi, \pi)$ will be zero. This is because determination of these coefficients are obtained by integrating

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |\sin(x)| \sin(nx) \, dx$$

and the fact that $\sin(nx)$ is an odd function and the product of an even and odd function is again odd, tells us that this integral will be zero, regardless of the value of n.