Homework Set # 2 – Math 435 – Summer 2013

1. Suppose that we have a uniform thin tube (approximable by one space dimension) of liquid with some particles which are suspended in the liquid. If the liquid is flowing through the pipe uniformly at a constant rate c (m/s), carrying with it the particles, and if we also take into account that the particles diffuse within the solution, derive the PDE for the concentration of the particles $u(x, t)$.

2. Suppose now that we have a motionless fluid in a tube, and again have particles suspended in that liquid. The particles move by diffusion AND sediment out of the solution at a fixed percentage rate v (in units 1/s - v is the fraction of particles that fall out of solution per second). Derive the PDE modeling the concentration of the particles $u(x, t)$.

3. Suppose that a uniform rod (approximable as one-dimensional) has a uniform heat source, so that the basic equation describing heat flow within the rod is

$$u_t = \alpha^2 u_{xx} + 1$$

for $0 \leq x \leq 1$. Suppose we fix the boundaries' temperatures so that at $x = 0$ the rod is held at temperature 0 and at $x = 1$ the rod is held at temperature 1.

   (a) Formulate the boundary conditions for the given problem.

   (b) Write the boundary value problem (meaning the PDE and the boundary conditions) that describes the steady-state temperature of the rod.

   (c) Use ODE techniques to solve the steady-state problem, if possible.

4. (a) What is a physical interpretation of the initial-boundary-value problem:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \qquad \text{for } 0 \leq x \leq 1 \ , \qquad 0 < t < \infty \\ u(0, t) &= 0 \\ u_x(1, t) &= 1 \qquad \text{for } 0 < t < \infty \\ u(x, 0) &= sin(\pi x) \qquad \text{for } 0 \leq x \leq 1 \end{aligned}$$

   (b) Can the solution come to a steady state? [hint: try to find steady-state solutions]

   (c) Answer (a) and (b) again, but with the boundary conditions

$$\begin{aligned} u_x(0, t) &= 0 \\ u_x(1, t) &= 0 \qquad \text{for} 0 < t < \infty \end{aligned}$$

5. (a) What is a physical interpretation of the initial-boundary-value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad \text{for } 0 \leq x \leq 1 \ , \qquad 0 < t < \infty \\ u(0, t) &= 0 \\ u(1, t) &= \sin(t) \qquad \text{for } 0 < t < \infty \\ u(x, 0) &= 0 \\ u_t(x, 0) &= 0 \qquad \text{for } 0 \leq x \leq 1 \end{aligned}$$

   (b) Can the solution come to a steady state?

6. Section 1.5 Strauss, problem 5

7. Section 1.5, problem 6

8. What are the types of the following equations (elliptic, parabolic, or hyperbolic)?

   (a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$

   (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

   (c) $u_{xx} - 4u_{xy} + 4u_{yy} = 0$

   (d) $u_{xx} - 4u_{xy} - 4u_{yy} = 0$

9. Section 1.6, Problem 2.

10. Use the rotational change of variables:

$$x = \xi \cos\theta - \eta \sin\theta$$
$$y = \xi \sin\theta + \eta \cos\theta$$

or equivalently:

$$\xi = x \cos\theta + y \sin\theta$$
$$\eta = -x \sin\theta + y \cos\theta$$

for some angle of rotation θ, to show that any equation of the form $au_{xx} + au_{yy} + bu = 0$ is invariant under rotation (the form of the equation doesn't change under the change of variables!).