1. (a) [5 points] Is $u_x + xu_y = 0$ linear? Either way, prove that your claim is true.

yes since $\mathcal{J}(u) := u_x + x u_y$ implies

$$\mathcal{J}(u+v) = (u+v)_x + x(u+v)_y = u_x + v_x + x u_y + x v_y = (u_x + x u_y) + (v_x + x v_y)$$
$$= \mathcal{J}(u) + \mathcal{J}(v)$$

and $\mathcal{J}(cu) = (cu)_x + x(cu)_y = c u_x + c x u_y = c(u_x + x u_y) = c\mathcal{J}(u)$

(b) [5 points] Is $u_x + u_y + 1 = 0$ homogeneous? What is it's order? Why?

No it is inhomogeneous, since writing it in standard form gives $u_x + u_y = -1$, so that the RHS is not zero.

It's order is 1 since the highest order derivative that appears is a first order derivative.

(c) [5 points] Show that $u(x, y) = 9x^2 + 6xy + y^2$ is a solution to the PDE $u_x - 3x_y = 0$.

$$u_x = 18x + 6y$$
$$u_y = 6x + 2y$$

$$\Rightarrow u_x - 3u_y = (18x + 6y) - 3(6x + 2y) = 18x + 6y - 18x - 6y = 0$$

and so u is a sol'n to the PDE.

(d) [5 points] Show that if $u_1, u_2, \ldots, u_n$ are all solutions to $L(u) = 0$, then $v = c_1u_1 + \cdots + c_nu_n$ is also a solution for any choice of constants c_i, for all i.

$u_1, u_2, \ldots, u_n$ are all solution to $L(u) = 0$, meaning $L(u_i) = 0$ for all $i = 1, \ldots, n$.

If $v = c_1 u_1 + \cdots + c_n u_n$ for some constants c_i, then since L is linear:

$$L(v) = L(c_1 u_1 + \cdots + c_n u_n) \quad \text{(by def of } v\text{)}$$
$$= c_1 L(u_1) + c_2 L(u_2) + \cdots + c_n L(u_n) \quad \text{(since } L \text{ is linear)}$$
$$= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0 \quad \text{(since } L(u_i) = 0 \ \forall i\text{)}$$
$$= 0$$

thus $L(v) = 0$ is also true regardless of the values of the c_i's //.

2. (a) [12 points] Find the general solution for $3u_t + 4u_x = 0$. If we add in the condition that $u(x,0) = e^{-x^2}$, what is the resulting ~~PDE well-posed~~ solution? Explain how this examples illustrates why the PDE is refered to as a "simple transport equation", and give the transport speed.

$3u_t + 4u_x = 0 \iff \langle 3, 4\rangle \cdot \langle u_t, u_x\rangle = 0$

$\iff$ level curves have slope $\frac{dx}{dt} = \frac{4}{3}$ at every x, t

$\iff$ the level curves are $x = \frac{4}{3}t + C$, or $x - \frac{4}{3}t = C$

$\iff$ the general solution $u(x,t)$ is $u(x,t) = f(x - \frac{4}{3}t)$.

Now if $u(x,0) = e^{-x^2} \Rightarrow u(x,0) = \underline{f(x) = e^{-x^2}}$, and so

$u(x,t) = e^{-(x-\frac{4}{3}t)^2}$. This illustrates why this is a "simple transport" model because the solution is just the initial profile e^{-x^2} translated to the right at speed $\frac{4}{3} = c$. Thus we can imagine it as the original concentration profile being transported by a fluid moving at constant speed $c = 4/3$.

(b) [8 points] Is the following system of equations well-posed? Explain.

$$u_x + 2xy^2 u_y = 0$$
$$u(x,0) = \phi(x)$$

Solving, we have:

$\langle 1, 2xy^2\rangle \cdot \langle u_x, u_y\rangle = 0$

$\Rightarrow \frac{dy}{dx} = 2xy^2$ for level curves of u.

$\Rightarrow \int \frac{1}{y^2}dy = \int 2x\,dx$ for level curves of u

$\Rightarrow -\frac{1}{y} = x^2 + C$ are the level curves of u

(i.e. $-\frac{1}{y} - x^2 = C$)

$\Rightarrow$ the general solution has the form

$u(x,y) = f(-\frac{1}{y} - x^2)$

When we try to apply the "initial" condition:

$u(x,0) = \phi(x) = f(-\frac{1}{0} - x^2)$

$\uparrow_!$

we see that we cannot.

Thus we cannot show existence of a sol'n for this problem $\Rightarrow$ ill-p

3. Consider the PDE

$$u_{xx} - 4u_{xy} - 4u_{yy} = 0$$

(a) [6 points] Classify the PDE as either elliptic, hyperbolic, or parabolic.

(b) [14 points] Choose a change of variables and use it to see the PDE as one of the wave equation/heat equation/Laplace's equation in terms of those new variables.

(a) $a_{11} = 1$, $a_{12} = -2$, $a_{22} = -4$ $\Rightarrow$ $a_{11}a_{22} = -4$, $a_{12}^2 = 4$ $\Rightarrow$ $a_{12}^2 > a_{11}a_{22}$ so the PDE is hyperbolic.

(b) Since the PDE is hyperbolic, we know there is some change of variables that will transform it into a PDE like the wave equation.

Try:

$$x = a_{11}\xi = \xi$$

$$y = a_{12}\xi + \sqrt{a_{12}^2 - a_{11}a_{22}}\,\eta = -2\xi + \sqrt{8}\,\eta$$

$$\Rightarrow \frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$$

$$\& \Rightarrow \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} = \sqrt{8}\frac{\partial}{\partial y}$$

"Factoring" our PDE by completing the square:

$$0 = \left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial y} - 4\frac{\partial^2}{\partial y^2}\right)u = \left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x\partial y} + 4\frac{\partial^2}{\partial y^2}\right)u - 8\frac{\partial^2}{\partial y^2}u$$

$$= \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)^2 u - 8\frac{\partial^2 u}{\partial y^2}$$

$$= \left(\frac{\partial}{\partial \xi}\right)^2 u - \left(\frac{\partial}{\partial \eta}\right)^2 u = u_{\xi\xi} - u_{\eta\eta}$$

↑ the wave eq'n in ξ, η.

4. (a) [7 points] Suppose that there is a function $u(x,t)$ such that $u_t = 3u_{xx}$ and that:

$$u(x,0) = 1 - x \qquad u_x(0,t) = 0 \qquad u(1,t) = 10$$

Describe fully one physical situation that could be modeled by this IBVP, and be sure to include what $u(x,t)$ represents as part of this description.

This could model the spread of heat in a uniform metal rod of length 1, where $u(x,t)$ = temperature of the rod at position x, time t. Then the initial temp. distribution would be described by $1-x$, the end at $x=0$ would be perfectly insulated, and the temp at $x=1$ would be held at zero degrees.

(b) [8 points] Assuming the time-dependent setup from part (a), can the physical system come to a steady state solution? If so, what is it and how is the result interpreted in the physical application? Does it match your physical intuition of the situation? If it cannot come to steady-state, why not?

The steady state is given by: $\begin{cases} 0 = 3U_{xx} \\ U'(0) = 0 \\ U(1) = 10 \end{cases}$ where $U = U(x)$.

$\Rightarrow U_{xx} = 0$, so $U(x) = Ax + B$.

$\rightarrow U'(x) = A$, so if $U'(0) = 0 \Rightarrow A = 0$.

Thus $U(x) = B$.

If $U(1) = 10 \Rightarrow U(1) = B = 10 \Rightarrow U(x) = 10$ for all x, or the temp the rod is a constant 10 degree

Physical Intuition: Since one end is perfectly insulated and the other is held at 10°, I expect the only solution unchanging over time to be $= 10°$ over the whole rod. Otherwise, if for example there is a region on the rod colder than 10°, the heat will conduct down the rod from the 10° end to warm it up... Similarly if there is a hotter region.

(c) [5 points] Suppose that you have a 10 inch long string undergoing small vibrations in a plane, one end of which is pinned down and the other end is begin moved by a machine so that it's height is given by $\sin(2\pi t)$ at any time t. If the string is initially at equillibrium, set up the IBVP that models this situation.

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(10,t) = \sin(2\pi t) \\ u(x,0) = 0 \end{cases}$$

5. [20 points] **Choose one:**

(a) Derive the wave equation from first principles. Include explanation for steps AND all assumptions made.

OR

(b) Derive the diffusion equation from first principles. Include explanation for steps AND all assumptions made.

<u>Wave equation</u>: Suppose we have a perfectly flexible elastic string of constant density ρ vibrating in a single plane (transversely <u>only</u>) such the amplitude of the vibration is very small. Let $u(x,t)$ = displacement from equilibrium at pos. x, tim

Let $\vec{T}(x,t)$ be the tension of the string at position x and time t.

Since it's flexible + elastic $\Rightarrow$ the tension will be <u>tangent</u> to the string itself, so

$$\vec{T}(x,t) = \|\vec{T}\| \frac{\langle 1, u_x(x,t)\rangle}{\sqrt{1+(u_x(x,t))^2}}$$ since $\langle 1, u_x\rangle$ is also always tangent to the string.

The amplitude of the vibrations are small $\Rightarrow$ u_x^2 is small

$$\Rightarrow \quad 1+u_x^2 \approx 1. \Rightarrow \vec{T} \approx \langle \|\vec{T}\|, \|\vec{T}\| u_x(x,t)\rangle$$

Since there is no vibration <u>longitudinally</u>, the net force in the x direction across any segment of string must be zero — so on the segment between $[x_0, x_1]$,

Since the force is due solely to the tension, we see

$$\Rightarrow \underbrace{\|\vec{T}(x_0,t)\|}_{\text{x component of tension at } x_0} - \underbrace{\|\vec{T}(x_1,t)\|}_{\text{at } x_1} = 0$$ $\Rightarrow$ the magnitude of the tension is the same at <u>every</u> point on the string, since x_0, x_1 are arbitrary.

Let $T = \|\vec{T}\|$, so that T is constant.

transversely, we have Newton's law $F = ma$ gives

net force in the y-direction on $[x_0, x_1]$ $= T u_x(x_1,t) - T u_x(x_0,t) = \displaystyle\int_{x_0}^{x_1} \rho u_{tt}\, dx$

Differentiating w.r.t. x_1, gives:

$$T u_{xx}(x_1,t) = \rho u_{tt}(x_1,t).$$

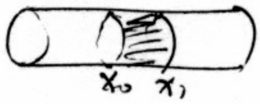
x_1 arbitrary $\Rightarrow u_{tt} = \frac{T}{\rho} u_{xx}$ for all x,t, or $u_{tt} = c^2 u_{xx}$ for all x,t with $c = \sqrt{\frac{T}{\rho}}$.

~~diff~~/heat equation:

Suppose we have a uniform thin tube of length l (approximable by 1D) containing a motionless fluid, within which there are particles of some substance suspended that move solely due to diffusion. Let $u(x,t)$ = concentration (gm/cm) at position x, time t of the substance.

Then the total mass of substance between x_0 + x_1 is

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,dx.$$

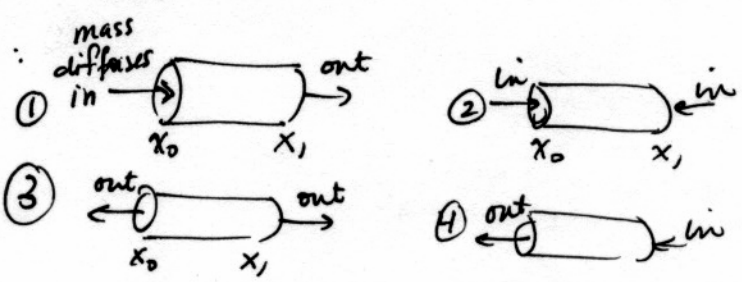


Thus the rate of change of the mass between x_0 + x_1 is, at any time t,

$$\frac{dM}{dt} = \frac{d}{dt}\left(\int_{x_0}^{x_1} u(x,t)\,dx\right) = \int_{x_0}^{x_1} u_t(x,t)\,dx. \qquad (1)$$

Thinking physically, the only way the amount of mass between x_0+x_1 can change is if some of it diffuses in or out through an end (at x_0 or x_1). Fick's Law says that the rate of diffusion across a point is proportional to the conc. gradient at that point, so the rate at which mass will cross each end is proportional to u_x at that end.

There are 4 possible cases:



for case ①, if mass diffuses in at x_0, we get a positive contribution to $\frac{dM}{dt}$ at this end (b/c mass inside is increasing). This occurs only if $u_x(x_0,t) \le 0$ since particles will diffuse from regions of higher to lower conc, and at the other end since we lose mass at x_1, we have $u_x(x_1,t) \le 0$ and this end gives a neg. contribution to $\frac{dM}{dt}$. By Fick's law + these observations:

$$\frac{dM}{dt} = k u_x(x_1,t) - k u_x(x_0,t)$$

In fact for all [redacted] cases, we have

$$\frac{dM}{dt} = ku_x(x_1, t) - ku_x(x_0, t). \qquad (2)$$

Putting (1) and (2) together, we see

$$\int_{x_0}^{x_1} u_t(x,t)\,dx = k u_x(x_1, t) - k u_x(x_0, t)$$

and differentiating w.r.t. x_1

$$\Rightarrow \quad u_t(x_1, t) = k u_{xx}(x_1, t).$$

Since x_1 is arbitrary $\Rightarrow$ $u_t = k u_{xx}$ for all x, t //