2. Let r be the radius and h be height of a right circular cone. The volume of the cone is given by $V = \frac{\pi r^2 h}{3}$.

(a) [10 points] Use the linear approximation to $V(r, h)$ at $r = 2$ and $h = 1$ to estimate the volume of the cone with $r = 2.08$ and $h = 0.99$.

$$L(r,h) = V_r(2,1)(r-2) + V_h(2,1)(h-1) + V(2,1)$$

$V_r = \frac{2\pi r h}{3}$

$V_h = \frac{\pi r^2}{3}$

$$= \frac{4\pi}{3}(r-2) + \frac{4\pi}{3}(h-1) + \frac{4\pi}{3}$$

So

$$V(2.08, 0.99) \cong L(2.08, 0.99) = \frac{4\pi}{3}(0.08) + \frac{4\pi}{3}(-0.01) + \frac{4\pi}{3}$$

actual volume when $r=2.08$, $h=0.99$; approx equal to ; linear approx to V at $(2,1)$ for $r=2.08, h=0.99$

$$= \frac{4\pi}{3}(1.07)$$

(b) [8 points] If we can only measure r and h up to 0.1 cm accuracy, use differentials to approximate the maximum possible error in the value of the volume when $r = 2$ and $h = 1$?

$$dV = V_r(2,1)\,dr + V_h(2,1)\,dh$$

$$\Rightarrow dV = \frac{4\pi}{3}dr + \frac{4\pi}{3}dh$$

to maximize dV, we take $dr + dh$ as big as possible, which from the problem, we see max $dr = 0.1$, max $dh = 0.1$

$$\Rightarrow dV = \left(\frac{4\pi}{3}\right)(0.1) + \left(\frac{4\pi}{3}\right)(0.1) = \frac{0.8\pi}{3}$$ ← approx. max error in volume.

(c) [10 points] Suppose now that r and h are changing over time by

$$r(t) = t^2 + t \quad \text{and} \quad h(t) = 1/t\,.$$

Then how fast is the volume changing when $t = 1$?

$$\frac{dV}{dt} = ?$$

by chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} + \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$\frac{dV}{dr} = \frac{2\pi r h}{3}$ $\qquad \frac{dr}{dt} = 2t+1$

$\frac{dV}{dh} = \frac{\pi r^2}{3}$ $\qquad \frac{dh}{dt} = -\frac{1}{t^2}$

when $t=1$:

$\frac{dr}{dt} = 3$ & $r(1) = 1^2+1 = 2$

$\frac{dh}{dt} = -1$ & $h(1) = \frac{1}{1} = 1$

$$\frac{dV}{dt}(t=1) = \frac{dV}{dr}(2,1)\frac{dr}{dt}(1) + \frac{dV}{dh}(2,1)\frac{dh}{dt}(1)$$

$$= \frac{4\pi}{3}(3) + \frac{4\pi}{3}(-1) = \boxed{\frac{8\pi}{3}}$$