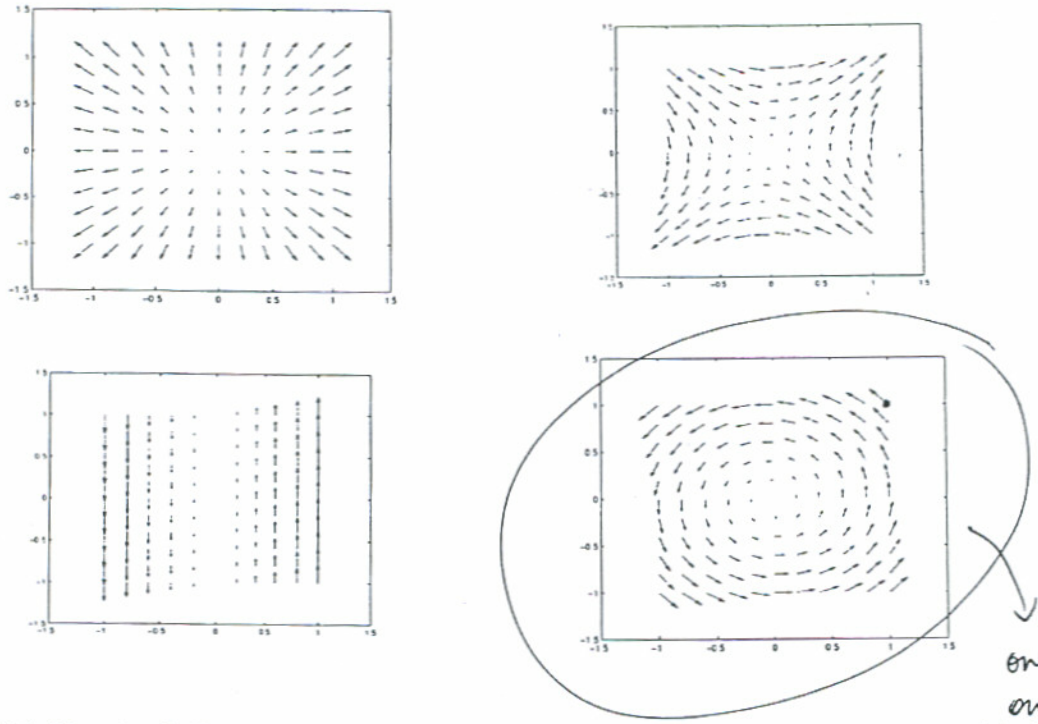
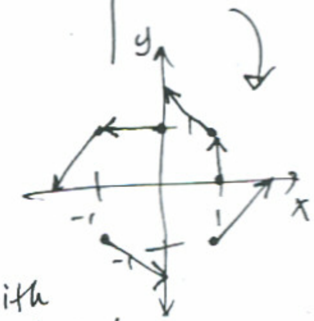
1. Let $\vec{F}(x,y) = < -y, x >$ snd $\vec{r}(t) = \langle 3-t, t^2+2 \rangle$.

(a) [8 points] Match the vector field with it's picture. Explain your choice in a way that rules out the other choices.



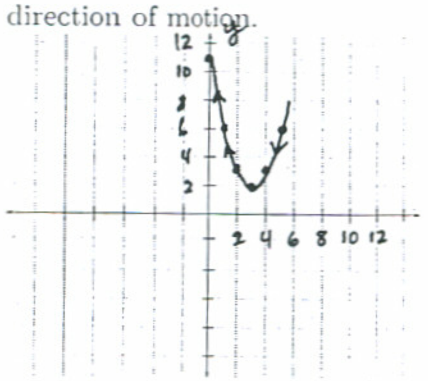
$\vec{F}(x,y) = \langle -y, x \rangle$

(x,y)	$\vec{F}(x,y)$
$(0,1)$	$\langle -1, 0 \rangle$
$(1,1)$	$\langle -1, 1 \rangle$
$(-1,1)$	$\langle -1, -1 \rangle$
$(1,-1)$	$\langle 1, 1 \rangle$
$(-1,-1)$	$\langle 1, -1 \rangle$
$(1,0)$	$\langle 0, 1 \rangle$



only one with correct pattern + correct vector @ $(1,1)$.

(b) [8 points] Sketch the curve $\vec{r}(t)$. Include at least 6 points and be sure to label the direction of motion.



t	$\vec{r}(t)$
0	$\langle 3, 2 \rangle$
1	$\langle 2, 3 \rangle$
-1	$\langle 4, 3 \rangle$
2	$\langle 1, 6 \rangle$
-2	$\langle 5, 6 \rangle$
3	$\langle 0, 11 \rangle$

(c) [10 points] Find the work done by the field $\vec{F}$ on an object moving along $\vec{r}(t)$ for $0 \le t \le 2$.

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_0^2 \langle -(t^2+2), 3-t \rangle \cdot \langle -1, 2t \rangle \, dt$$

$$= \int_0^2 t^2 + 2 + 6t - 2t^2 \, dt = \int_0^2 -t^2 + 6t + 2 \, dt$$

$$= -\frac{1}{3}t^3 + 3t^2 + 2t \Big|_0^2 = -\frac{8}{3} + 12 + 4$$

2. Let $\vec{F}(x,y) =< y\cos(xy)+1, x\cos xy + 8y >$.

(a) [14 points] Show $\vec{F}$ is conservative and find a potential f.

domain of $\vec{F}$ is $\mathbb{R}^2$, which is open and simply connected, so we need only check that $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ to know $\vec{F}$ is conservative:

$$\frac{\partial F_2}{\partial x} = \cos(xy) - x\sin(xy)(y) = \cos(xy) - xy\sin(xy)$$

$$\frac{\partial F_1}{\partial y} = \cos(xy) - y\sin(xy)(x) = \cos(xy) - xy\sin(xy) \checkmark \Rightarrow \vec{F} \text{ conservative}$$

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if $\vec{\nabla} f = \vec{F} \Rightarrow \frac{\partial f}{\partial x} = y\cos(xy)+1$ and $\frac{\partial f}{\partial y} = x\cos(xy) + 8y$

$\Rightarrow f(x,y) = \int y\cos(xy)+1 \, dx$

let $u = xy$ $\Rightarrow du = y\,dx$ $\rightarrow = \int \cos u \, du + x$

$= \sin(xy) + x + C(y)$

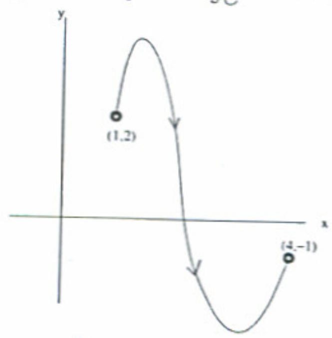
$\Rightarrow \frac{\partial f}{\partial y} = x\cos(xy) + C'(y)$

$= x\cos(xy) + 8y$

$\Rightarrow C'(y) = 8y \Rightarrow C(y) = 4y^2 + C$

$\Rightarrow \boxed{f(x,y) = \sin(xy) + x + 4y^2 + C}$

(b) [8 points] Find $\int_C \vec{F}\cdot d\vec{r}$ for both curves shown below. Explain your answers.



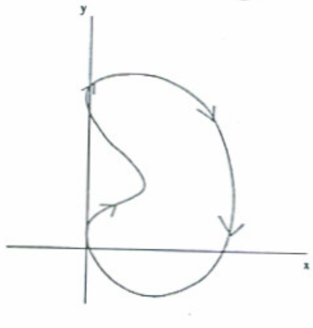
Since $\vec{F}$ is conservative, by (a), we can use the fundamental theorem of line integrals:

$$\int_C \vec{F}\cdot d\vec{r} = f(4,-1) - f(1,2)$$

$$= [\sin(-4) + 8 + C]$$

$$- [\sin(2) + 17 + C]$$

$$= \sin(-4) - \sin(2) - 9$$



Since this is a closed loop, the beginning and end points are the same point, so the fund. thm gives

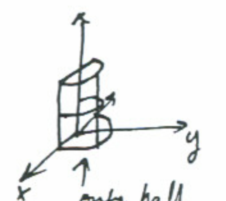
$$\int_C \vec{F}\cdot d\vec{r} = 0.$$

4. Let $f(x, y) = -2x^2 + 4xy + 2y^2$

(a) [5 points] Give a parametrization for $f(x, y)$.

$$\vec{r}(u,v) = \langle u, v, -2u^2 + 4uv + 2v^2 \rangle$$

(b) [10 points] Find the surface area of the part of $f(x, y)$ that lies inside the cylindar $x^2 + y^2 = 9$ and such that $y \geq 0$.



only half w/ $y \geq 0$

$0 \leq \theta \leq \pi$

$0 \leq r \leq 3$

$$S.A. = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$\vec{r}_u = \langle 1, 0, -4u + 4v \rangle$

$\vec{r}_v = \langle 0, 1, 4u + 4v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 4u - 4v, -4u - 4v, 1 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{16u^2 - \cancel{16uv} + 16v^2 + 16u^2 + \cancel{16uv} + 16v^2 + 1}$

$= \sqrt{32u^2 + 32v^2 + 1}$

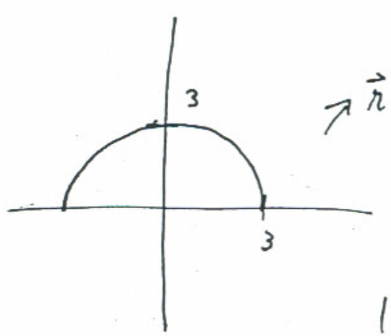
$$= \iint_D \sqrt{32u^2 + 32v^2 + 1} \, du \, dv$$

$$= \int_0^{\pi} \int_0^3 \sqrt{32r^2 + 1} \; r \, dr \, d\theta$$

let $u = 32r^2 + 1 \Rightarrow du = 64r \, dr$

$$= \frac{1}{64} \int_0^{\pi} \int_0^3 \sqrt{u} \, du \, d\theta = \frac{2}{3} \cdot \frac{1}{64} (32r^2 + 1)^{3/2} \Big|_0^3 \cdot \int_0^{\pi} d\theta = \frac{\pi}{96} \left[(32 \cdot 9 + 1)^{3/2} - 1 \right]$$

(c) [10 points] Let the curve C be the semicircle in the xy-plane of radius 3 with $y \geq 0$. Compute $\int_C f \, ds$. What does this represent? it represents the signed area under f and over C.



$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$

$0 \leq t \leq \pi$

$\vec{r}'(t) = \langle -3\sin t, 3\cos t \rangle$

$|\vec{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t} = 3$

$$\int_C f \, ds = \int_0^{\pi} (-18\cos^2 t + 36\cos t \sin t + 18\sin^2 t) \cdot 3 \, dt$$

$$= 48 \int_0^{\pi} -\cos^2 t + \sin^2 t + \cos t \sin t \, d$$

$$= 48 \int_0^{\pi} (-\cos(2t) + \cos t \sin t) \, dt$$

$$= -\frac{48 \sin(2t)}{} \Big|_0^{\pi} + \frac{48 \sin^2 t}{2} \Big|_0^{\pi} = 0$$

Polar Coordinates: $x = r\cos\theta$ $y = r\sin\theta$
$x^2 + y^2 = r^2$

Spherical Coordinates: $x = \rho\cos\theta\sin\phi$ $y = \rho\sin\theta\sin\phi$ $z = \rho\cos\phi$ $\rho^2 = x^2 + y^2 + z^2$

Cylindrical Coordinates: $x = r\cos\theta$ $y = r\sin\theta$ $z = z$
$x^2 + y^2 = r^2$

Arclength and Curvature For a curve $\vec{r}(t)$ defined for $t \in [a, b]$, the arclength of the curve is given by

$$\int_a^b \left|\frac{d\vec{r}}{dt}\right| dt$$

Also, the curvature $\kappa(t)$ of the curve $\vec{r}$ is

$$\kappa(t) = \frac{|\frac{d\vec{T}}{dt}|}{|\frac{d\vec{r}}{dt}|} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$