Solutions to Selected Problems:

Section 3.2

(1) Let $x(t)$ represent the AMOUNT of salt in the tank at any given time t. Then $\frac{dx}{dt}$ represents the rate of change of the amount of salt in time (units kg/min). According to the problem statement, the salt will be coming into the tank at a rate of $(8L/min) * (.05kg/L) = 0.4kg/min$ and is exiting at a rate of $(\frac{x}{100}kg/L) * (8L/min) = \frac{8x}{100}$ kg/min. Thus, the ODE for x is

$$\frac{dx}{dt} = .4 - \frac{8x}{100}$$

This ODE is linear, so we can take the integrating factor $e^{8t/100}$ and multiply through by it. Recognizing the product rule on the left hand side, we get

$$(e^{8t/100}x)' = .4e^{8t/100}$$

so that

$$e^{.08t}x = 5e^{8t/100} + c$$

and $x(t) = 5 + ce^{-8t/100}$. Since $x(0) = .5$, we have $5 + c = 0.5$ or $c = -4.5$. Thus $x(t) = 5 - 4.5e^{-.08t}$. Now to find when $x = 0.02 * 100 = 2kg$, we solve

$$2 = 5 - 4.5e^{-0.08t}$$

which results in

$$5.07 \text{ min} = \frac{\ln(2) - \ln(3)}{-0.08} = t \; .$$

(5) If we have a solution that is .001% chlorine, then we have by volume $\frac{.00001 \text{ gal chlorine}}{\text{gal solution}}$. So, if again x is the amount of chlorine in the pool, measured in gallons, the incoming rate of chlorine is (.00001 gal Cl/ gal solution) $*$ 5(gal solution/min) = .00005 gal Cl/min. The outgoing rate is $x/10000$ (gal Cl/gal solution)$*$5 gal solution/min $= \frac{5x}{10000}$ gal Cl/min. So the ODE is

$$\frac{dx}{dt} = .00005 - .0005x$$

which is linear and can be solved via the integrating factor $e^{.0005t}$.

We get

$$x(t) = .1 + ce^{-.0005t} \; .$$

Applying the inital condition that the pool originally has .01% chlorine, meaning it has $.0001*10000 = 1$ gallon of chlorine initally, gives us

$$x(t) = .1 + .9e^{-.0005t} \; .$$

Letting $t = 60$ minutes gives us that $x(60) = .9734$ gallons of chlorine, which means its concentration in the pool is .9734/10000, which means its percentage in the pool is $100 * (.9734/10000) = .9734/100 = .009734\%$.

3.3

(9) The average outside temp is $\frac{32+16}{2} = 24$, so the model curve for M is

$$M(t) = 24 - 8\cos(\pi t/12) \ .$$

Here we assume that $t = 0$ corresponds with 2 am and $t = 12$ corresponds with 12 hours later at 2 pm. Since there is no heating/cooling system (U(t)=0) and the ambient contributions to the temperature are not taken under consideration (H(t) = 0), we get the model for the change in temperature is

$$\frac{dT}{dt} = K[24 - 8\cos(\pi t/12) - T] \ .$$

This is linear and can be solved to obtain

$$T(t) = 24 - 8\left(\frac{\cos(\pi t/12) + \frac{pi}{12K}\sin(\pi t/12)}{1 + \frac{\pi^2}{12^2K^2}}\right) + Ce^{-Kt} \ .$$

If the time constant for the building is 1, then $K = 1$. If the time constant is 5, then $K = 1/5$. We can then use this solution to find the maximum and minimum temperatures depending on K. The max and min occur at critical points, or when $T' = 0$. Thus we look at

$$T' = \frac{8}{1 + \pi^2/(12^2K^2)} * \left(-\frac{pi}{12}sin(\pi t/12) + \frac{\pi^2}{12^2K}\cos(\pi t/12)\right) = 0$$

and we get

$$\tan(\pi t/12) = \frac{\pi}{12K} \quad \rightarrow \quad t = \frac{12}{\pi}\arctan(\frac{\pi}{12K}) \ .$$

If $K = 1$ then a critical temp occurs for $t = .978radians$ which gives the minimum temp of $T = 16.26°C$. Since tangent has period π, another critical temp occurs at $t = .978 + 12$. Evaluating T at this time should give the maximum possible temperature. Note: you could use a graphing calculator to see the max and min temps... this is probably the simplest way to get them.

If $K = 1/5$ then a critical temp occurs for $t = 3.508$, which gives the minimum temp of $T = 19.14°C$. Again, the max temp should occur for $t = 3.508 + 12$ (and should be lower than that of $K = 1$ since this reflects a better insulated situation).

(13) We begin by remembering that $\frac{dT}{dt}$ = heat in - heat out. Heat is added to the tank via the solar panel. Heat is lost due to the difference between tank temp and outside temp. So the model is

$$\frac{dT}{dt} = (2°F/1000Btu) * (2000Btu/hr) + (1/64)[80 - T]$$

or

$$\frac{dT}{dt} = 4 + (1/64)[80 - T] \ .$$

This equation is linear, and can be solved as such. The general solution is

$$T(t) = 64 * 4 + 80 + Ce^{-t/64} = 336 + Ce^{-t/64} \ .$$

Applying the initial condition that $T(0) = 110$, we find $C = -226$, so that the temp in the tank after 12 hr of sunlight is

$$T(12) = 336 - 226e^{-12/64}$$