Ex: find general sol'n for

$$y'' + 4y = \sec^4(2t)$$

homog. sol'n: $y_h = c_1\cos(2t) + c_2\sin(2t)$

nonhomog: $y_p = v_1\cos(2t) + v_2\sin(2t)$

with: $v_1'\cos(2t) + v_2'\sin(2t) = 0$

and $-2v_1'\sin(2t) + 2v_2'\cos(2t) = \sec^4(2t)$

$$\Rightarrow \quad 2v_2' = \frac{1}{\cos^3(2t)}$$

$$v_2' = \frac{1}{2\cos^3(2t)}$$

$$v_2 = \frac{1}{2}\int \sec^3(2t)\,dt = \frac{1}{2}\int \sec(2t)\cdot\sec^2(2t)\,dt$$

let $\begin{cases} u = \sec(2t) \\ dV = \sec^2(2t)\,dt \\ du = +2(\cos(2t))^{-2}\sin(2t) \\ \quad = 2\sec(2t)\tan(2t) \\ V = \frac{1}{2}\tan(2t)\,dt \end{cases}$

$$\Rightarrow = \frac{1}{4}\sec(2t)\tan(2t) - \frac{1}{2}\int \sec(2t)\tan^2(2t)\,dt$$

$$= \frac{1}{4}\sec(2t)\tan(2t) - \frac{1}{2}\int \frac{\sin^2(2t)}{\cos^3(2t)}\,dt$$

$$= \frac{1}{4}\sec(2t)\tan(2t) - \frac{1}{2}\int\left(\frac{1}{\cos^3(2t)} - \frac{1}{\cos(2t)}\right)dt$$

$$= \frac{1}{4}\sec(2t)\tan(2t) - \frac{1}{2}\int \sec^3(2t)\,dt + \frac{1}{2}\int \sec(2t)\,dt$$

$$\Rightarrow \int \sec^3(2t)\,dt = \frac{1}{4}\sec(2t)\tan(2t) + \frac{1}{2}\ln\left|\sec(2t) + \tan(2t)\right|$$

so $\boxed{v_2 = \frac{1}{8}\sec(2t)\tan(2t) + \frac{1}{8}\ln|\sec(2t)+\tan(2t)|}$

Now: $v_1' = \frac{-v_2'\sin(2t)}{\cos(2t)} = -\frac{1}{2}\frac{\sin(2t)}{\cos^4(2t)}$

$$\Rightarrow v_1 = \frac{1}{4}\cdot\left(-\frac{1}{3}\right)\cos^{-3}(2t) = -\frac{1}{12}\sec^3(2t)$$

$$y_p = -\frac{1}{12}\sec^2(2t) + \frac{1}{8}\tan^2(2t) + \frac{1}{8}\sin(2t)\ln|\sec(2t)+\tan(2t)|$$

$$\Rightarrow \boxed{y = c_1\cos(2t) + c_2\sin(2t) - \frac{1}{12}\sec^2(2t) + \frac{1}{8}\tan^2(2t) + \frac{1}{8}\sin(2t)\ln|\sec(2t)+\tan(2t)|}$$

this is the same as the result in the back of the book

Since $\dfrac{\cos^2(2t) + \sin^2(2t) = 1}{\cos^2(2t)} \Rightarrow 1 + \tan^2(2t) = \sec^2(2t)$

$$\Rightarrow \quad 1 = \sec^2(t) - \tan^2(2t)$$

$$\Rightarrow -\frac{1}{8} = -\frac{1}{8}\sec^2(2t) + \frac{1}{8}\tan^2(2t)$$

$$-\frac{1}{8} = \left(-\frac{1}{12} - \frac{1}{24}\right)\sec^2(2t) + \frac{1}{8}\tan^2(2t)$$

$$\Rightarrow -\frac{1}{8} + \frac{1}{24}\sec^2(2t) = -\frac{1}{12}\sec^2(2t) + \frac{1}{8}\tan^2(2t)$$

so

$$\boxed{\begin{aligned} y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{8} + \frac{1}{24}\sec^2(2t) \\ + \frac{1}{8}\sin(2t)\ln\left|\sec(2t) + \tan(2t)\right| \end{aligned}}$$