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Math 231:Introduction to Ordinary Differential Equations
Mini-Project: When Zombies Attack

Brains!! Braaaaiiinnss!! I lie awake at night dreading the day those words are uttered in flat, loud, hungry tones by the undead in my near vicinity. I am sure you are no less fearful. Fortunately for us all, a 2009 paper called When zombies attack! Mathematical modelling of an outbreak of zombie infection helps us solve the age old problem of what to do when in fact zombies attack.

Let S represent the number of people (the susceptibles), Z is the number of zombies, and R (the removed) represents (a) deceased zombies, (b) bitten people (who are sometimes turned into zombies), or (c) dead people - i.e. R represents everyone who is not a living susceptible or (not-so)living zombie. We develop a differential equation model that is similar in spirit to the ones in the paper mentioned above. Upon some good old-fashioned internet research it seems that not everyone is in agreement on how people can turn into zombies. Will begin with a model that follow the assumptions that a human can turn into a zombie if bitten by a zombie OR a human can turn into a zombie by some unknown process through which it is ressurected from the dead, though it died without any kind of zombie virus and there is no interaction with a zombie necessary.

$$\frac{dS}{dt} = BS(1 - S/K) - aSZ - dS \tag{1}$$

$$\frac{dZ}{dt} = aSZ + cR - eSZ \tag{2}$$

$$\frac{dR}{dt} = eSZ + dS - cR \tag{3}$$

$$\tag{4}$$

where we assume that people have a net birth/death rate $B - d$ when the population is far from the carrying capacity of their environment. Some of the interactions between a person and a zombie result in a person killing a zombie by destroying its brain - the fraction of the total interactions that are in this class is given by e. Some of these interactions results in a zombie infecting a person and turning that person into a zombie - the fraction of interactions in this class is given by a. Finally, we have some fraction c of the otherwise normal dead humans resurrecting as zombies.

The second model will explore rests a slightly different assumptions about how humans become zombies. We have

$$\frac{dS}{dt} = rS(1 - S/K) - aSZ \tag{5}$$

$$\frac{dZ}{dt} = aSZ + cRS - eSZ \tag{6}$$

$$\frac{dR}{dt} = eSZ - cRS \tag{7}$$

$$\tag{8}$$

In some zombie story circles, it is said that even dead zombies can transmit the zombie virus to a human through contact with either the zombie corpse or it's fluids, BUT that a human that died from other means cannot ressurect as a zombie. To account for transmission to humans through contact with dead zombies that are being removed to be buried or otherwise disposed of, we have the interaction term cSR with which we model the that dead zombies are removed (buried, or something similar) so that they are no long able to come into contact with humans, but that very removal process can transmit the disease and turn more humans into zombies. Here r is the net birth-death rate in the absence of zombies for the human population.

You will use these models to investigate what it would take for humankind to survive a zombie attack! For a three person group, the intention here is that the first and last problems will be approached together, while each of 2,3, and 4 will be under the responbility of one group member.

1. First, we will use the first model and assume that $B = 0$, so that no one is born during the time this model is being applied. Show that $\frac{d(S+Z+R)}{dt} = 0$ for all time. Explain why this tells us that our total population being modeled really will stay constant, and we can solve for R in terms of S and Z (explicitly), so that we really need only the ODE's for S and Z in order to know the full dynamics of this system over time. Write down this system in only S and Z. We will assume for the remainder of these problems that the total population of susceptible, zombie, and "recovered" is always 500.

   Next, repeat the above process for the second model, assuming $r = 0$.

2. Do a full phase plane analysis for the system from number 1, with parameters $a = 0.5$, $d = 0.01$, $B = 0$, $c = 0.1$, $e = 0.3$. What do these parameters say about the "situation on the ground"? Note any equillibria and what outcome they represent. What are all of the possible long term outcomes in this scenario?

   Next, use three different sets of initial values $(S(0), Z(0))$ from different regions of the phase plane, if possible, and use MATLAB to solve the system with those initial conditions. Is there any hope for mankind?

   Now, since it seems likely to be the coefficient that is the most in humanities control to change, let's investigate how the parameter e affects our outcomes. If we increase e so that $e = 0.6$, what does this represent? Redo the steps above with this new value for e, holding all other parameters the same. What changes, if any do you see in the possible long-term outcomes? Can you explain why these changes occured, if there are any? Or why there is no change?

   Would it help at all to take even larger values for e? Explain why or why not.

3. For this part, we will use the second model where do we not have spontaneously ressurecting zombies from the dead, and we also assume that there are no births or deaths due to other causes during the time period we are modeling (so $r = 0$). Use the same coefficients as for the first part of the second problem. Again, do a full phase plane analysis, noting the equillibria. Now, choose three different sets of initial values $(S(0), Z(0))$ and use MATLAB to solve the system using those initial conditions. If possible, take the initial conditions from different regions of your phase plane. Again, interpret your results in terms of the zombie attack.

   Now, since it seems likely to be the coefficient that is the most in humanities control to change, let's investigate how the parameter e affects our outcomes. If we increase e so that $e = 0.6$, what does this represent? Redo the steps above with this new value for e, holding all other parameters the same. What changes, if any do you see in the possible long-term outcomes? Can you explain why these changes occured, if there are any? Or why there is no change?

   Compare your results for this model to your results in problem 2 and explain why these differences in results arise from the different choice of terms in the model, if there are any differences. If there are no differences in possible long term outcomes, explain why that is.

Again, does it help to take even larger values for e? Why or why not?

4. In this part we'll investigate the effect of the "c terms" in each model on the outcomes for humanity by eliminating those terms in each model and comparing the results to the results from parts 2 and 3.

   First, use the first system of differential equations. Use coefficients $a = 0.5$, $e = 0.6$, $d = 0.01$, $B = 0$, $c = 0$. Do a full phase plane analysis, noting the equillibria, and listing all possible long term outcomes. Now, choosethree different sets of initial values $(S(0), Z(0))$ and use MATLAB to solve the system using those initial conditions. If possible, take the initial conditions from different regions of your phase plane. Again, interpret your results in terms of the zombie attack.

   Compare your results for this model to your results in problem 2 and explain why these differences in results arise from absence or present of the c terms, if there are any differences, in terms of the difference in the situation on the ground. If there are no differences in possible long term outcomes, explain why that is.

   Finally, repeat the experiment with $c = 0$ for the second system of differential equations, using the same coeffcients as above, with $r = 0$.

5. Notice that in the prior problems, we've assumed there are no births occuring in the human population. Now, let's take $B = .01$ for the first system,or $r = 0.01$ for the second system and $K = 500$, using the values of the coefficients with $e = 0.6$ and $a = 0.5$, $c = 0.1$, and $d = 0.01$. Notice that when we add the three equations together now, we no longer get that the derivative of the sum of the three populations $S(t)$, $Z(t)$ and $R(t)$ is zero. Thus, we can no longer assume the total population is constant. Since this is the case, we cannot do the phase plane analysis for this part.

   Instead, just solve the system for at least three different initial conditions on MATLAB, making sure you try well to determine whether or not there is more than one possible long term outcome. Interpret your results.

   Does adding in births change the hope for mankind? If not, does it help if you take some higher birth rates B and r ? What about if you increase K? Does love in fact save the day? Or do we just end up supplying the world with MORE zombies?

   Summarize your findings from all of the problems above. Given this information, what emergency plan would you put in place in the event of a Zombie Attack?