Department of Mathematics Fall 2012
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Math 231:Introduction to Ordinary Differential Equations Mini-Project: Romeo & Juliet, the dynamics of affection

Ah, love. The single most powerful force in the universe, some believe. Romantic relationships in particular can be ecstatic and difficult, bringing out the best and worst in each of us. With a topic as important and central to our daily lives as this, how could we neglect to mathematically model it? Maybe we can gain some insight into creating more love in our lives through the results? Let's see!

We will pretend here, to model Romeo and Juliet's young love. Passionate and narcissistic, reactive, all the things adolescent love should be ;). Let's look at a few modeling assumptions, their corresponding models, and the resulting relationship dynamics that ensue. The results for each individual model should be the primary responsibility of one team member each and the last question summarizing the results should be answered as a group.

It is very important that you all get together and make sure you understand what each term in the models represents, and what the coefficients "measure", before trying to interpret your results.

Model I First, let us assume that Romeo is head over heals in love with Juliet, and he is also insecure. The more she loves him, the more he loves her, but if she shows displeasure with him, he reduces his affection for her. Juliet on the other hand too is crazy about Romeo. However, she is in love with being in love, so that more she loves him, her own love for him inspires even more love for him. The catch is that, as the cliche goes, if Romeo's love for her is too much she begins to like him less, and if he shows dislike for her, she likes him more. We can model this by letting $R(t)$ represent the level of Romeo's love for Juliet and $J(t)$ the opposite. Here R or J negative indicates dislike.

$$R'(t) = aJ \tag{1}$$

$$J'(t) = bJ + (c - dR) \tag{2}$$

Instructions:

1 Sort out what each term of the system of differential equations models in terms of how these two tend to behave in relationship. What would each coefficient give the weight of?

Let's begin by letting $c = 15$, $a = 1$, $b = .5$, and $d = .5$. Do the full phase plane analysis for this problem, noting any equillibria. What are the possibilties for what happens with this relationship over time? Does it matter where their affection levels start out?

2 Solve the system by hand for $J(t)$ and $R(t)$, leaving a, b, c and d unfixed (so as undetermined constants, rather than with a given value plugged in). You should see that there are two different possibilities for the forms of each solution $J(t)$ and $R(t)$ depending on whether or not $b^2 - 4ad$ is positive or negative. What are those possible forms?

Now use MATLAB to plot at least three different solution sets $R(t)$ and $J(t)$ for different initial values, each from a different region of the phase plane. Use the coefficient values as set in part (1). What do you notice? Use your solutions $J(t)$ and $R(t)$ with the values of the coefficients set as in part (1) plugged in to determine a specific solution form. Make sure that the functions you obtain for $R(t)$ and $J(t)$ agree with the behavior exhibited by your MATLAB solutions.

3 Finally, Suppose now that we model Juliet's feelings as we did originally, but Romeo's feelings are **constant**, so that the system becomes

$$\frac{dR}{dt} = 0$$

$$\frac{dJ}{dt} = bJ + (c - dR)$$

What must be true for Juliet's love to decrease? When will it increase?

Solve for $J(t)$ if $R(t) = 10$ for all time, so that $\frac{dJ}{dt} = bJ + (c - 10d)$. How Juliet feels about Romeo over time now depends only on how she feels about him initially. Show what happens to the solution for the intial condition that $J(0) = 0$, and what happens for the intial condition that $J(0) = -22$. What exactly determines whether our outcome is happiness or unhappiness in the long-run?

Repeat for Romeo's constant affection to be still constant but larger, so $R(t) = 20$ for all time. What differences do you see and why do those differences occur in terms of the changes in their emotional responses?

Model II Let's change up the dynamics of their feelings a bit and see what happens. Suppose now that things are bit less cliche, and a bit more... fair. Now Romeo is also "in love with being in love", so that the more he loves her, it encourages him to love her that much more, and he also loses interest when she's too enamored.

$$R'(t) = aR + (e - J)R \tag{3}$$

$$J'(t) = bJ + (c - R)J \tag{4}$$

Instructions: Construct the full phase plane for this new system with $a = 1$, $e = 15$ and $b = 1$, $c = 15$ so that they are exactly mirrors of each other in terms of their emotional response. Explain what possibilities we have now for how this relationship can evolve.

This system cannot be solved by hand, but you can solve the phase plane equation for this system. So, solve the phase plane equation, leaving a, b, e, and c as arbitrary constants, finding the general form of any phase plane trajectory for this system. What happens to the solution if $R(0) = J(0)$ and what would that represent as far as their feelings are concerned?

Now, use MATLAB to find approximate solutions to the system with the choice of coefficients given above. Find at least three different solutions that begin with initial conditions chosen from at least three different regions of the phase plane. Make sure to choose at one initial condition where $R(0) = J(0)$. Update your conclusions about what the possible long term outcomes are for this relationship.

Now let's make them a little bit different from one another by taking $a = 1$ and $e = 15$, but $b = .5$ and $c = 20$. Repeat the steps above for this system. Are there any differences in the long term outcomes or in the *likelihood* of any of the outcomes? Explain any differences you see in terms of the changes in the relationship shown by the changes in the coefficients.

Model III Suppose that both Romeo and Juliet have aged and experienced too much heartbreak. They both now have a limit to how much they are willing to love someone else who is showing no interest in them, and we'll represent the measure of this limit by the values of K_1 (Romeo's limit) and K_2 (Juliet's limit). Let's suppose also that now they no longer think someone else can love them too much, and how they feel about things depends primarily on how in sync their feelings are (this might be a rough measure of how much each feels that the other understands them). If g_1 and g_2 represent a weighing of how much being out of sync/in sync affects their feeling for the other person, and if s_1 and s_2 represent a weighing of how responsive one is to their own feelings about the other person, we can model the changes in their emotions for one another by

$$R'(t) = g_1 JR + s_1 R(1 - \frac{R}{K_1}) \tag{5}$$

$$J'(t) = g_2 JR + s_2 J(1 - \frac{J}{K_2}) \tag{6}$$

Find the full phase plane for this model by hand, letting $K_1 = K_2 = 15$, $s_1 = 2$, $s_2 = 0.5$, and $g_1 = .5$, $g_2 = 2$. What do these coefficient choices tell us about the relationship? Looking at the resulting phase plane, what are the possible long-term outcomes? What seems to be the deciding factor as to whether or not they will be happy long-term?

Use MATLAB to approximate the solutions $J(t)$ and $R(t)$ over time for at least three different interesting initial conditions taken from different regions in your phase plane.

Next, find the full phase plane with $g_1 = -0.5$ and all other parameters the same. What does this change in g_1 represent in terms of Romeo's emotional response to the relationship? Looking at the new phase plane, what are the possible long term outcomes now? Now is there any deciding factor that determines whether or not they will be happy long-term?

Use MATLAB again so approximate the solution for at least three interesting initial conditions chosen from different regions of your phase plane if possible.

IV Summarize the what the results tell us from these three models about relationships. Are there ideal behaviors that lead to a "best" outcome, according to your results?

If you were to build your own model of a romantic relationship, what features from above would you include? Which would you leave out? Why? What features would you like to add that are not discussed above? How could you mathematically model them?

As a group, design one more (simple,but not too simple :)) model and use MATLAB to explore some of the long-term outcomes. Explain the results you get in terms of the ideas you included in creating the model and the assumptions you made about their emotional responses to one another.