Department of Mathematics
The University of Tennessee

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H. Finotti

Math 231:Introduction to Ordinary Differential Equations
Mini-Project: Price vs. Quantity Economic Model

Consider the following economic model: Let P be the price of a single item on the market. Let Q be the quantity of the item available on the market. Both P and Q are functions of time. The following model was proposed in "A First Course in Mathematical Modeling" by Giordano, Weir, and Fox.

$$\frac{dQ}{dt} = cQ(fP - Q) \tag{1}$$
$$\frac{dP}{dt} = aP\left(\frac{b}{Q} - P\right) \tag{2}$$

We could recognize both of these equations as a kind of logistic growth model for first P and the Q respectively. In the first, we see the "carrying capacity" for P depends on the quantity of items on the market and is given by $\frac{b}{Q}$, and in the second we see that the "carrying capacity" for Q depends is given by fP and so is dependent upon the price of the object.

The former implies that the more items on the market, the less we'll be able to charge for them, which reflect a common principle of supply and demand, that the more of an item that is available in the market place, likely the lower the demand will become. Lower demand for our item would likely lead the manufacturer to lower the price of the item in order to increase demand for it.

The latter implies that the higher the price of the item the more we can feel free to have in the market place. This balances the manufacturers need to profit from the item with wanting to follow the consumer demand in pricing. If the price is low, there is not as much money to be made from our items, which has implications for manufacturing of those items for the future.

This model then at first glance captures this kind of push and pull between pricing to meet demand and pricing to earn profit. Let's analyze the model and see how well it does in the end. Together as a group do problems 1 and 5. Each member of a three person group will be responsible for one of problems 2, 3, or 4 individually.

1. Explain what each of the coefficients a, b, c, and f represent in terms of the economics of the system. Explain what the model implies for $P = 0$ and for Q very small or very large. Are these implications realistic? Of those that are justify why they are, and of those that aren't explain why you feel they aren't.

2. If $a = 1$, $b = 20000$, $c = 1$, and $f = 30$, create a full phase plane for the system and note the equillibria. What are the potential long-term outcomes for this system? What do they represent in terms of the economic situation?

   Next, use MATLAB to approximate solutions to this system for at least three different initial conditions chosen from three different regions of the phase plane, if possible. Interpret your results.

   In order to investigate the role of a in the system, next let $a = 5$ and repeat the steps above, with all other coefficients remaining the same. What do you notice? Are there any qualitative differences between the dynamics of the system in this case versus the case of $a = 1$? Explain why if there are and why not if there aren't.

3. Now we'll investigate the role of f in the system. First let $f = 100$ and repeat the steps for the first part of problem 2, leaving all other coefficient values the same as in that first part of problem 2. What do you notice? Are there any qualitative differences between the dynamics of the system in this case versus the case of $f = 30$? Explain why if there are and why not if there aren't, in terms of the economic meaning of f and the dynamics of the price/quantity system.

   Next, instead let $f = 15$ and repeat the steps again. What do you notice? Are there any qualitative differences between the dynamics of the system in this case versus the case of $f = 30$? Explain why if there are and why not if there aren't, in terms of the economic meaning of f and the dynamics of the price/quantity system.

   What general conclusions would you draw based on your work about how f affects the long-term price structure for the item? Are your results consistant with what f represents economically? Explain.

4. Finally, we'll investigate the role of b in the system. First let $b = 5000$ and repeat the steps for the first part of problem 2, leaving all other coefficient values the same as in that first part of problem 2. What do you notice? Are there any qualitative differences between the dynamics of the system in this case versus the case of $b = 20000$? Explain why if there

are and why not if there aren't, in terms of the economic meaning of b and the dynamics of the price/quantity system.

Next, instead let $b = 40,000$ and repeat the steps again. What do you notice? Are there any qualitative differences between the dynamics of the system in this case versus the case of $b = 20000$? Explain why if there are and why not if there aren't, in terms of the economic meaning of b and the dynamics of the price/quantity system.

What general conclusions would you draw based on your work about how b affects the long-term price structure for the item? Are your results consistant with what b represents economically? Explain.

5. As a group, construct the full phase plane for this system leaving a, b, c, and f as arbitrary constants. What are the coordinates of the equillibria in terms of these coefficients? According to you prior work, what do you expect to see long-term? Illuminate which of these coefficients determines the long-term outcome values for the price and quantity of items, and which don't have any effect. Explain why this is in economic terms. For those that don't effect the long-term price/quantity outcomes, how *do* they effect the dynamics of the system?

   One modification of this model that might work just as well would be to have the $\frac{dQ}{dt}$ equation the same as above, but replace the $\frac{dP}{dt}$ equation with

$$\frac{dP}{dt} = aP\left(\frac{bQ}{Q^2+1} - P\right) .$$

   Do you see any obvious advantages to using this equation for P rather than the first? Any disadvantages?