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Math 231:Introduction to Ordinary Differential Equations
Mini-Project: Microparasites and their Hosts

Simple mathematical models for microparasites offer a useful way to examine the population dynamics of different viral and bacterial pathogens. One constraint in applying these models in freeliving host populations in the lack of data with which to estimate transmission rates. DeLeo and Dobson showed that hosts having different body sizes suffer epidemic outbreaks whose frequency scales with body size. The resulting epidemic periods for pathogens in different mammalian populations given by the model correspond to the cycles observed in free-living populations when taking body size into account!

We'll look at two models that take into account different features of the physical situation. Let $S(t)$ be size of the population of a specific animal that is susceptible to the parasite and let $I(t)$ be the size of the population of that same animal that is infected with the parasite. (All content of this project is excerpted from the *Science* article by DeLeo and Dobson.)

$$\frac{dS}{dt} = (v - \mu)[1 - \left(\frac{S+I}{K}\right)]S - \beta SI \tag{1}$$

$$\frac{dI}{dt} = \beta SI - (\alpha + \mu)I \tag{2}$$

Here v represents the average birth rate, μ the natural mortality rate, α the disease induced mortality rate, β rate of transmission of the disease, and K is the carrying capacity of the environment.

Several of these parameters depend on the body mass of the species being affected. We will take $K = 16.2w^{-0.7}$, $\mu = 0.4w^{-0.26}$, $v = 0.6w^{-0.27}$, $\alpha = (m-1)*\mu$, and $\beta = 0.4*m*w^{-0.26}$. We cam see here that our value of m acts as an indication of the how deadly a particular disease is, in that it multiplies the natural death rate to give us the death rate due to disease. Note that it also shows up in the transmission rate.

1. In this part, we'll explore the effect of w on the outcome of a disease. Let $w = 1$ kg first and $m = .05$. Do a full phase plane analysis of the system. What are the possible long-term outcomes? Choose at least three different initial conditions from interesting regions of your phase plane, and use MATLAB to approximate solutions to the system having those initial conditions. Plot the results for $S(t)$ and $I(t)$ over time. What do you see?

   Next, increase $w = 60$ and repeat. Are the long term possible outcomes changed or not? Again, look at several approximate solutions via MATLAB and plot them over time. It would be interesting to use the same initial conditions that you used before, so you can draw comparisons, as well as some added in to capture any new regions of phase space. Is there any qualitative change in the dynamics of the disease?

2. Now let's investigate the effect of m on the outcome of a disease. Let's take again $w = 1$ kg, and let $m = .01$. Do a full phase plane analysis of the system. What are the possible long-term outcomes? Choose at least three different initial conditions from interesting regions of your phase plane, and use MATLAB to approximate solutions to the system having those initial conditions. Plot the results for $S(t)$ and $I(t)$ over time. What do you notice in terms of differences between this case and the case for $m = 0.05$ with the same w value? Explain any differences you see in terms of the physical changes in the system reflected by the change in parameter.

   Next, increase to $m = .1$ keeping our weight the same at 1 kg and repeat. Are the long term possible outcomes changed or not? Again, look at several approximate solutions via MATLAB and plot them over time. It would be interesting to use the same initial conditions that you used before, so you can draw comparisons, as well as some added in order to capture any new regions of phase space (if necessary). Is there any qualitative change in the dynamics of the disease? Explain any differences you see again in physical terms.

3. Now let's change the model a bit. Let's take instead:

$$\frac{dS}{dt} = (v - \mu)[1 - \left(\frac{S + I}{K}\right)]S - \beta SI \tag{3}$$

$$\frac{dI}{dt} = \beta SI - \left(\alpha + \mu + (v - \mu)\left(\frac{S + I}{K}\right)\right) I \tag{4}$$

   where our parameters are as above. This is called a "density-dependent mortality" model. Explain why this name is relevant.

We'll again investigate the role of species size (modeled by weight w) in the outcomes for this disease. Repeat the experiments for number 1, using this model instead.

3.5 This part is only to be done if the group has a fourth member. For this part, use the density-dependent mortality model from number 3, but this time fix $w = 1$ and investigate the effect of changing the value of m, just as was done in number 2.

4. Summarize your findings above. What sort of hypothesis can you formulate about how microparasitic disease dynamics differ between large and small animals? Does it seem to matter how virulent a disease is in how the dynamics unfold? How does density-dependent mortality effect your hypothesis, if at all?