Department of Mathematics Fall 2015
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Math 231:Introduction to Ordinary Differential Equations
Mini-Project: the dynamics of the heartbeat

The heart is a complicated but robust pump. It has four chambers and four valves. There are two circuits for the blood, one which spreads through the lungs to pick up oxygen and the other which spreads through the body to deliver the oxygenated blood. The first circuit is a low-pressure one so as not the damage the delicate membrane in the lungs, and the second is a high=pressure circuit so that the blood gets all the way down to the feet and up again.

During the heart beat cycle there are two extreme equilibrium states, namely diastole which is the relaxes state and systole which is the contracted state. What make the heart beat is the presence of a pacemaker which islocated on the top of the atrium. The pacemaker causes the heart to contract into systole. That is, it triggers off an electtrochemical wave whcih spreads slowly over the atria causing the muscle fibers to contract and push blood into the ventricles, and then spreads rapidly over the ventricles, causing the whole ventricle to contract into systole and deliver a big pump of blood down the arteries. The muscle fibers then rapidly relax and return the heart to a diastole; the process is then repeated.

In order to develop a mathematical model which reflects the behavior of the heart beat action described above, we choose to single out the following features. First of all, the model should exhibit an equilibrium state corresponding to diastole. Secondly, there must be a threshold for triggering the electrochemical wave emanating from the pacemaker causing the heart to contract into systole. Thirdly, the model must reflect the rapid return to the equilibrium state.

Let x represent the muscle fiber length in reference to some convenient origin $x = 0$ that represents the equilibrium state. Let b be an electrical control variable which governs the electrochemical wave that initiates the fiber contraction. A model which incorporates the desired features is:

$$\epsilon \frac{dx}{dt} = -(x^3 - ax + b) \tag{1}$$

$$\frac{db}{dt} = x - x_a \tag{2}$$

Here a represents the muscle tension, b represents the chemical control and x_a is the typical fiber length when the heart is in diastole. This model is due to E.C. Zeeman.

This is one of the earliest and simplest models of the heartbeat. It is not the only model, nor is it the most complete model. BUT it has been quite successful in distinguishing between some extreme forms of heart beat behavior, like the effects of high blood pressure or an excess of adrenaline in the bloodstream due to rage or vigorous exercise. Likewise, there is the situation when the heart is by-passed during an operation, in which case the heart beats in a feeble manner and does not contract into systole.

My intention for a group of three would be that each member of the group takes responsibility for one of problems 2, 3 or 4. Problems 1 and 5 should be done as a group.

1. Let $\epsilon = 0.001$, $a = .81$, and $x_a = .45$. Do a full phase plane analysis for the system. For this system, it is actually important when drawing the phase plane direction arrows to notice the vastly different scale for b' and x'. Because of ϵ being so small, x' will actually be quite alot larger than b', so the direction of the arrow is dominated by x' and for at least some of the regions b' is negligible. For this reason when you sketch your phase plane, make sure you indicate clearly those reasons where the arrows are quite large in magnitude due to a large value of x'. Note any equillibria, and try to sketch some trajectories in the phase plane by hand. Do at least three of them, each starting from a different point in the phase plane where you think the ensuing dynamics might be interesting.

   Using three different initial conditions from different regions of the phase plane if possible, use MATLAB to plot the solutions $x(t)$ and $b(t)$ over time in each case. Make sure to try and choose inital conditions that will potentially have different long term outcome. I think it can be particularily interesting throughout this project to take at least one initial condition that is close to an equillibrium point (but not AT it).

2. Let's investigate the impact that changing the tension a has on the behavior of our system. It's important to realize that in the case of someone with high blood pressure, the heart gets overworked and the fibers stretched, so that the tension in the heart muscle is reduced, and the heart is not able to pump as efficiently. Take a couple of smaller values for a, $a = .6$ and $a = .4$ and redo number 1. Again, make sure that you try to choose inital conditions for your MATLAB investigations that have the potential to give you different outcomes long term.

   What does this tell us about the potential effects of high blood pressure?

3. If we significantly increase the tension, what happens? Try values of $a = 4$ and $a = 10$ and redo number 1. What are the differences between what you see now and what the results were in problems 1 and 2?

   Explain what this says about the potential effects of high tension in the heart. It is not shown directly through your results, but do you see a mechanism by which high tension can lead to stretched heart fibers and low blood pressure?

4. Investigate the role that the value of ϵ plays in the behavior of the solution to the equations. What happens to the phase plane and the MATLAB solutions to the system when you decrease ϵ by a factor of 10? What happens when you increase it to $\epsilon = 1$? Note the differences you see in the dynamics for each case as compared the the base case in part 1 where $\epsilon = 0.001$. In these investigations, keep all other coefficient values the same as in part 1.

   It is hard to give a physiological meaning to ϵ. It is here, really to force the model to behave in the way that we want. Which one of the criteria in the introduction to this project that we wanted our heart model to meet does ϵ help the model achieve? It may be instructive to run several MATLAB simulations for various values of ϵ of your choosing, letting it get very large or very small. I

5. Summarize what the problems 1, 2, 3, and 4 tell us physiologically about the heart.