#9 – Section 3:

Have a min temp $= 16°C$ (at 2am.) and max $= 32°C$ (at 2pm.) outside. We know $\frac{1}{k} \in [1,5]$.

Model outside temp by

$$M(t) = 24 + 8\cos\left(\frac{\pi t}{12}\right)$$

← has max $= 32$, min $= 16$ and period $\frac{2\pi}{\pi/12} = 24$ hrs.

$$\Rightarrow \frac{dT}{dt} = k\left(24 + 8\cos\left(\frac{\pi t}{12}\right) - T\right)$$

or

$$\frac{dT}{dt} + kT = k\left(24 + 8\cos\left(\frac{\pi t}{12}\right)\right)$$

let $\mu(t) = e^{\int k\,dt} = e^{kt} \Rightarrow \frac{d}{dt}\left(e^{kt}T\right) = ke^{kt}\left(24 + 8\cos\left(\frac{\pi t}{12}\right)\right)$

So

$$e^{kt}T = \int ke^{kt}\left(24 + 8\cos\left(\frac{\pi t}{12}\right)\right)dt$$

$$= 24e^{kt} + 8k\int e^{kt}\cos\left(\frac{\pi t}{12}\right)dt$$

(★ – see next page!)

$$= 24e^{kt} + 8k\left[\left(\frac{12^2k^2}{\pi^2 + 12^2k^2}\right)\left[\frac{e^{kt}}{k}\cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{k^2 12}e^{kt}\sin\left(\frac{\pi t}{12}\right)\right]\right] + C$$

$$\Rightarrow \boxed{T(t) = 24 + \frac{(8)(12^2k^2)}{\pi^2 + 12^2k^2}\left[\cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12k}\sin\left(\frac{\pi t}{12}\right)\right] + Ce^{-kt}}$$

(*) Work for finding

$$\int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt$$

let $u = \cos\left(\frac{\pi t}{12}\right) \Rightarrow du = -\frac{\pi}{12} \sin\left(\frac{\pi t}{12}\right)$

$dv = e^{kt}$ $\quad v = \frac{1}{k} e^{kt}$

so: $$\int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt = \frac{1}{k} e^{kt} \cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12k} \int e^{kt} \sin\left(\frac{\pi t}{12}\right) dt$$

$$\Rightarrow \int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt = \frac{1}{k} e^{kt} \cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12k} \left[\frac{1}{k} e^{kt} \sin\left(\frac{\pi t}{12}\right) - \frac{\pi}{12k} \int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt\right]$$

$$\Rightarrow \underline{\int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt} = \frac{1}{k} e^{kt} \cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12k^2} e^{kt} \sin\left(\frac{\pi t}{12}\right) - \underline{\frac{\pi^2}{12^2 k^2} \int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt}$$

adding this term to both sides.

$$\Rightarrow \left(1 + \frac{\pi^2}{12^2 k^2}\right) \int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt = \frac{1}{k} e^{kt} \cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12k^2} e^{kt} \sin\left(\frac{\pi t}{12}\right)$$

$$\Rightarrow \boxed{\int e^{kt} \cos\left(\frac{\pi t}{12}\right) dt = \frac{12^2 k^2}{12^2 k^2 + \pi^2} \left[\frac{1}{k} e^{kt} \cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12k^2} e^{kt} \sin\left(\frac{\pi t}{12}\right)\right] + C}$$

The max + min values of T will occur at critical points,

so look at $\frac{dT}{dt} = 0$, assuming the exponential term is negligible.

$$\Rightarrow \quad \frac{8(12^2k^2)}{\pi^2 + 12^2k^2}\left[-\frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right) + \frac{\pi^2}{12^2 k}\cos\left(\frac{\pi t}{12}\right)\right] = 0$$

$$\Rightarrow \quad \sin\left(\frac{\pi t}{12}\right) = \frac{\pi}{12k}\cos\left(\frac{\pi t}{12}\right) \Rightarrow \tan\left(\frac{\pi t}{12}\right) = \frac{\pi}{12k}$$

$$\text{or} \quad t = \frac{12}{\pi}\arctan\left(\frac{\pi}{12k}\right)$$

if $k=1$ (time const $=1$) $\Rightarrow$ $t = \frac{12}{\pi}\arctan\left(\frac{\pi}{12}\right) \approx 0.978$ is one critical point.

if $k = \frac{1}{5}$ (time const $=5$) $\Rightarrow$ $t = \frac{12}{\pi}\arctan\left(\frac{5\pi}{12}\right) \approx 3.508$ is one critical point.

we know all the critical points are solutions to

$$\tan\left(\frac{\pi t}{12}\right) = \frac{\pi}{12k}$$

since tangent is periodic with period π,

if t_0 is a critical value of t, then

$$\tan\left(\frac{\pi t_0}{12} + n\pi\right) = \frac{\pi}{12k} \quad \text{is also true, for any integer } n.$$

$$\text{or} \quad \tan\left(\frac{\pi}{12}(t_0 + 12n)\right) = \frac{\pi}{12k} \quad \text{is also true}$$

$\Rightarrow$ $t_0 + 12n$ are all critical values if t_0 is.

So to see the highest + lowest temps:

If $k=1$, $t_0 = 0.978$, $t_0 + 12 = 12.978$

and $T(0.978) = 31.74\ °C$

$T(12.978) = 16.26\ °C$

note these are almost the same as the max/min outdoor temps! $k=1$ represents [illegible] insulation.

if $k = \frac{1}{5}$: $t_0 = 3.508$, $t_0 + 12 = 15.508$

$$\Rightarrow T(3.508) = 24 + \left(\frac{8 \cdot 12^2}{25\pi^2 + 12^2}\right)\left[\cos\left(\frac{\pi(3.508)}{12}\right) + \frac{5\pi}{12}\sin\left(\frac{\pi(3.508)}{12}\right)\right]$$

$$= 28.86\,°C$$

$$T(15.508) = 24 + \left(\frac{8 \cdot 12^2}{25\pi^2 + 12^2}\right)\left[\cos\left(\frac{\pi(15.508)}{12}\right) + \frac{5\pi}{12}\sin\left(\frac{\pi(15.508)}{12}\right)\right]$$

$$= 19.14\,°C$$

So if the time constant for the building is 5, ($k = \frac{1}{5}$) the temp variation inside is from 19.14°C to 28.86°C while the temp outside varies from 16°C to 32°C. Thus this represents a situation with much more effective insulation.