MATH 567–PROBLEM SET 1–8/25/2011

1. Let $G \to Diff(M), g \mapsto \varphi_g$ be an action of a group G on a manifold M by diffeomorphisms. Consider the orbit separation conditions:

(A) $\forall p \in M$ and sequence $(g_n)_{n\geq 1}$, $\varphi_{g_n}(p)$ does not converge in M.

(B) $\forall p \in M \exists U$ open neighborhood of p in M so that $\varphi_g(U) \cap U = \emptyset, \forall g \in G$.

(C) $\forall p, q \in M$ in different G-orbits $\exists U, V$ open neighborhoods of p, q in M (resp.) so that $\varphi_g(U) \cap V = \emptyset, \forall g \in G$.

1.1 Show that condition (C) implies the quotient space $N = M/^{\sim}$ (with the quotient topology) is Hausdorff.

1.2 Consider the action of $\mathbb{Z}$ on $M = \mathbb{R}^2 \setminus \{0\}$ given by:

$$\varphi_n(v) = (2^{-n}v_1, 2^n v_2); \quad n \in \mathbb{Z}, v = (v_1, v_2).$$

Show that this action satisfies condition (B), but not condition (C). (*Hint:* for $v \in M$, consider a box $(v_1 - \epsilon, v_1 + \epsilon) \times (v_2 - \epsilon, v_2 + \epsilon)$, and compute how it maps under ϕ_n. For the second part, consider in particular boxes V at $(0,1)$ and U at $(1,0)$.

1.3 Find an example of a group action satisfying (C), but not (B). (*Hint:* consider rational rotations of the circle.)

1.4 Show that if M has a metric d (inducing its topology) and the action is by *isometries*, then condition (A) implies both (B) and (C).

2. Grassmannians and linear algebra. Denote by $G_{n,m}$ the set of n-dimensional vector subspaces of $\mathbb{R}^m$, $1 \leq n < m$. In this problem we show $G_{n,m}$ can be given the structure of a compact smooth manifold.

Let $F_{n,m} \subset M_{n\times m}$ be the set of $n \times m$ matrices A of full rank ($rk(A) = n$). This is an open set in the euclidean space $M_{n\times m} \sim \mathbb{R}^{mn}$, as the complement of the intersection of finitely many zero sets of $n \times n$ determinant minors.

We represent an n-dimensional subspace of $\mathbb{R}^m$ as the row space of a matrix in $F_{n,m}$; for this purpose, we identify two matrices if they have the same row space. Thus, introduce in $F_{n,m}$ the equivalence relation:

$$A \sim B \text{ iff } B = RA, \text{ for some } R \in GL_n.$$

The Grassmanian is then the quotient space: $G_{n,m} = F_{n,m}/\sim$ (with the quotient topology); denote by $\pi : F_{n,m} \to G_{n,m}$ the quotient projection.

2.1. Show that the quotient topology in $G_{n,m}$ is Hausdorff. *Hint:* to show the graph of the equivalence relation is closed, note that A and B have the same row space iff each of the n matrices of type $(n+1)\times m$ obtained by attaching to A successively each row of B has less than full rank.

To describe coordinate charts, we need the following notation. Let $I = (i_1, \dots, i_n)$ be a multi-index of length n with increasing entries, each an integer between 1 and m. Denote by I' the complementary index of length $m-n$, e.g. if $n=2, m=5$ and $I=(1,3)$ we have $I'=(2,4,5)$. Given I, define two maps on matrices:

$$P_I : M_{n\times n} \to M_{n\times n}, \quad Q_I : M_{n\times m} \to M_{n\times)m-n)};$$

$P_I(A)$ is obtained from A by keeping only the columns with indices in I, $Q_I(A)$ by only keeping the columns appearing in I'. From standard matrix algebra, given any $A \in F_{n,m}$ there exists a multi-index I so that $P_I(A) \in GL_n$, and then A is row-equivalent to a unique matrix $A_I \in F_{n,m}$ so that $P_I(A_I) = \mathbb{I}_n$, the identity matrix. In fact:

$$A_I = P_I(A)^{-1}A.$$

Conversely, given $C \in M_{n\times(m-n)}$ and a multi-index I, we can build a matrix in $F_{n,m}$ by placing the columns of $\mathbb{I}_n$ at the indices in I, and the columns of C at the indices in I'. Denote this matrix by $\tilde{\varphi}_I(C) := [\mathbb{I}|C]_I \in F_{n,m}$. Clearly then:

$$A_I = [\mathbb{I}|Q_I(A_I)]_I.$$

2.2. Let

$$\tilde{U}_I := \{A \in F_{n,m}; P_I(A) \in GL_n\}, \quad U_I = \pi(\tilde{U}_I) \subset G_{n,m}.$$

Both are open subsets, and define finite open coverings of $F_{n,m}$, resp. $G_{n,m}$. Consider the map:

$$\psi_I : U_I \to M_{n\times(m-n)}, \quad \psi_I([A]) = Q_I(A_I).$$

Claim: show that ψ_I is well-defined, and is a homeomorphism onto $M_{n\times(m-n)} \sim \mathbb{R}^{n(m-n)}$, with inverse given by:

$$\varphi_I : M_{n\times(m-n)} \to U_I, \quad \varphi_I(C) = \pi([\mathbb{I}|C]_I).$$

2.3. Given multiindices $I, \bar{I}$, the domains of coordinate changes are the open sets:

$$\tilde{\varphi}_I^{-1}(\tilde{U}_I \cap \tilde{U}_{\bar{I}}) = \varphi_I^{-1}(U_I \cap U_{\bar{I}}) := W_{I\bar{I}} \subset M_{n\times(m-n)}.$$

Claim: show that the coordinate changes:

$$\psi_{\bar{I}} \circ \varphi_I : W_{I\bar{I}} \to W_{\bar{I}I}$$

are diffeomorphisms (smooth homeomorphism with smooth inverse.)

3. (*Boothby*) Let $X = \mathbb{R} \times \{\pm 1\}$ with the product topology and smooth structure. Introduce an equivalence relation in X by $(x, i) \sim (y, j)$ if either (i) $x = y < 0$ or (ii) $x = y \geq 0$ and $i = j$. Show that the quotient space $X/\sim$ (with the quotient topology) is locally euclidean and has a countable basis of open sets but is not Hausdorff. Is the saturation of an open set in X open? Is the graph of the equivalence relation closed?