COMPLETELY INTEGRABLE GEODESIC FLOWS (two examples.)

1. Hyperbolic plane. $\mathbb{H} = (\{y > 0\}, ds^2 = \frac{dx^2+dy^2}{y^2})$.

Equations for geodesics. A fixed-endpoint variation of the Lagrangian for parametrized curves $\gamma(t) = (x(t), y(t))$:

$$L[\gamma] = \int_a^b \frac{\dot{x}^2 + \dot{y}^2}{y^2} dt$$

yields the second-order Euler-Lagrange equations:

$$\ddot{x} = 2\frac{\dot{x}\dot{y}}{y}, \qquad \ddot{y} = \frac{\dot{y}^2 - \dot{x}^2}{y}.$$

In the tangent bundle $T\mathbb{H}^2$, with coordinates (x, y, v, w), we have the equivalent first-order system:

$$\dot{x} = v, \quad \dot{y} = w, \quad \dot{v} = \frac{2vw}{y}, \quad \dot{w} = \frac{w^2 - v^2}{y}.$$

These can also be written in standard Lagrangian form:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial v}), \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial w}).$$

(The first equation gives the conservation of $\mathcal{K}$, which is equivalent to the equation for $\dot{v}$; the second equation gives $\dot{w}$; see below for the definitions of $\mathcal{L}$ and $\mathcal{K}$).

Conserved quantities. The following are conserved quantities for solutions of the first-order system:

$$\mathcal{L} = \frac{v^2 + w^2}{y^2}, \quad \mathcal{K} = \frac{v}{y^2}.$$

$\mathcal{L}$ is just the element of hyperbolic arc length. The geometric interpretation of $\mathcal{K}$ is obtained if we recall the expression for the *euclidean* curvature of the curve $\gamma(t) = (x(t), y(t))$:

$$k_e = \frac{\ddot{\gamma} \cdot \dot{\gamma}^\perp}{|\gamma|_e^3}, \quad \ddot{\gamma} = (\dot{v}, \dot{w}), \quad \dot{\gamma}^\perp = (-\dot{w}, \dot{v}), \quad |\gamma|_e^2 = v^2 + w^2 = \mathcal{L}y^2.$$

We have: $k_e = -\mathcal{L}^{-1/2}\mathcal{K}$. Thus the constancy of $\mathcal{K}$ and $\mathcal{L}$ along geodesics implies the constancy of the euclidean curvature: geodesics are contained

in arcs of circles (if $\mathcal{K} \neq 0$) or line segments (if $\mathcal{K} = 0$). As $y \to 0_+$ we must have $v \to 0$; so we have semicircles or half-lines intersecting the x-axis orthogonally.

Complete integrability. A vector field in $\mathbb{R}^n$ is *completely integrable* if computing its flow can be reduced to one-variable integration; equivalently, if it is equivalent to an uncoupled system of autonomous first-order ODE.

In the present case, from $v^2 + w^2 = y^2\mathcal{L}$ and $v^2 = y^4\mathcal{L}$ it follows thhat $w^2 = y^2(\mathcal{L} - \mathcal{K}y^2)$, so $\mathcal{L} \geq \mathcal{K}y^2$ and:

$$\dot{y} = \pm y\sqrt{\mathcal{L} - \mathcal{K}y^2}, \quad \int \frac{dy}{y\sqrt{\mathcal{L} - \mathcal{K}y^2}} = \pm dt.$$

(With y we find $v = \mathcal{L}y^2$, then x and w.)

Hamiltonian structure. The configuration space variables $q = (q_1, q_2) = (x, y)$ have "conjugate momenta":

$$p = (p_1, p_2), \quad p_1 = \frac{v}{y^2}, \quad p_2 = \frac{w}{y^2}.$$

Note $p_1 = \mathcal{K}$. With the Hamiltonian $H(q, p) = \frac{1}{2}q_2^2(p_1^2 + p_2^2)$ $(= \mathcal{L}/2)$, it is easy to check the first-order equations of motion can be written in the canonical form:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2} \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}.$$

Here $\dot{p_1} = -\frac{\partial H}{\partial q_1} = 0$ corresponds to the conservation of $\mathcal{K}$.

Poisson bracket. The conserved quantities $H(q, p) = (y^2/2)(p_1^2 + p_2^2)$ and $\mathcal{K}(q, p) = p_1$ are "in involution", that is, their *Poisson bracket* $\{H, \mathcal{K}\}$ vanishes:

$$\{H, \mathcal{K}\} = \sum_{i=1}^{2}\left(\frac{\partial H}{\partial p_i}\frac{\partial \mathcal{K}}{\partial q_i} - \frac{\partial \mathcal{K}}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = 0.$$

Exercise. Generalize the above to hyperbolic n-space. Compute the equations of geodesics in Lagrangian and Hamiltonian form, find n conserved quantities in involution and verify the complete integrability of the geodesic flow.

2 Surfaces of revolution. A parametrized (immersed) "profile curve" $((f(y), g(y))$ in $\mathbb{R}^2$ $(f(y) > 0, y \in \mathbb{R})$ gives rise to the (immersed) surface of

revolution in $\mathbb{R}^3$ parametrized by:

$$F(x,y) = (f(y)\cos(2\pi x), f(y)\sin(2\pi x), g(y)), \quad (x,y) \in T^1 \times \mathbb{R}, \quad T^1 = \mathbb{R}/\mathbb{Z}.$$

(We assume here y is the arc-length parameter along meridians, $(f')^2 + (g')^2 \equiv 1$.) The associated Lagrangian on curves $\gamma(t) = (x(t), y(t))$ in $T^1 \times \mathbb{R}$ is:

$$L[\gamma] = \int_a^b [f^2(y)\dot{x}^2 + \dot{y}^2]dt.$$

Euler-Lagrange equations. A variation with fixed endpoints gives the second order Euler-Lagrange equations for critical points of L:

$$\ddot{x} = -2\frac{f'}{f}(y)\dot{x}\dot{y}, \quad \ddot{y} = ff'(y)\dot{x}^2.$$

The equivalent first-order system in $(x, y, v, w) \in T(T^1 \times \mathbb{R})$ is:

$$\dot{x} = v, \quad \dot{y} = w, \quad \dot{v} = -2\frac{f'}{f}(y)vw, \quad \dot{w} = ff'(y)v^2.$$

Conserved quantities. The following are conserved under the first-order system:

$$\mathcal{L} = f^2(y)v^2 + w^2, \quad \mathcal{C} = f^2(y)v.$$

$\mathcal{L}$ is the element of arc length in the metric $ds^2 = f^2(y)dx^2 + dy^2$ and $\mathcal{C}$ is the *Clairaut invariant*. The first-order system in standard Lagrangian form is:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial v}), \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial w}).$$

These are equivalent to the first-order equations for $\dot{v}$ and $\dot{w}$, respectively.

Complete integrability. From:

$$\mathcal{L} = \mathcal{C}v + w^2 = \frac{\mathcal{C}^2}{f^2(y)} + w^2$$

we find:

$$\dot{y} = \pm\sqrt{\mathcal{L} - \frac{\mathcal{C}^2}{f^2(y)}}.$$

This can be integrated to yield $y(t)$, and then successively we find v (using $\mathcal{C}$) and $x(t)$.

Hamiltonian structure. The configuration variables $q = (q_1, q_2) = (x, y) \in T^1 \times \mathbb{R}$ have as "conjugate momenta":

$$p = (p_1, p_2), \quad p_1 = -f^2(y)v, \quad p_2 = w.$$

The Hamiltonian is:

$$H(q, p) = -\frac{p_1^2}{2f^2(q_2)} + \frac{p_2^2}{2}.$$

The first-order system written in canonical form is:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2} \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}.$$

Note that the equation $\dot{p_1} = \frac{\partial H}{\partial q_1} = 0$ is equivalent to the conservation of $\mathcal{C}$.

Poisson bracket. The conserved quantities $H(q, p)$ and $\mathcal{C}(q, p)$ are "in involution": $\{H, \mathcal{C}\} = 0$.

Exercise. Generalize this to $M = T^n \times \mathbb{R}$ with a "skew-product metric" (where $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-torus):

$$ds^2 = f^2(y)dx^2 + dy^2, \quad (x, y) \in T^n \times \mathbb{R},$$

where $f(y) > 0$ and dx^2 denotes the standard flat metric on T^n. Compute the equations of geodesics in Lagrangian and Hamiltonian form, find $n + 1$ conserved quantities in involution and verify the complete integrability of the geodesic flow.

Conclusion: Liouville integrability. These are two concrete examples of Liouville's complete integrability theorem, loosely stated: *a Hamiltonian system with n degrees of freedom is completely integrable if and only if it admits n independent conserved quantities "in involution".*

Here the conserved quantities $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are said to be *independent* if at each point of a (smooth) common level set $\{(q, p) | \mathcal{P}_i = c_i, i = 1, \ldots, n\}$ the gradient vectors $\nabla \mathcal{P}_i$ are linearly independent. The conserved quantities are *in involution* if the Poisson bracket of any two of them vanishes: $\{\mathcal{P}_i, \mathcal{P}_j\} = 0$.

Exercise. Show that in both examples the given conserved quantities are independent.

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