WHITNEY’S IMMERSION / EMBEDDING THEOREMS ¹

Preliminaries 1: Linear Algebra. Denote by $M(m \times n; k)$ the space of $m \times n$ matrices with real coefficients of rank $k$. Claim. This is a smooth surface in $M_{m \times n}$, of dimension $k(m + n - k)$, and therefore codimension $(m - k)(n - k)$.

Write $X \in M_{m \times n}$ in block form:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \ A \in M_{k \times k}. $$

Let $W = \{X \in M_{m \times n}; \det A \neq 0\}$. We have:

$$U := W \cap M(m \times n; k) = \{X \in W; D = CA^{-1}B\},$$

since the rank of $X \in M_{m \times n}$ is the same as the rank of:

$$\begin{bmatrix} I_k & 0 \\ -CA^{-1} & I_{m-k} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix},$$

and hence equals $k$ iff $D - CA^{-1}B = 0$. Parametrize $U$ by $\phi : U_0 \rightarrow U$,

$$U_0 = \{(A, B, C) \in M_{k \times k} \times M_{k \times (n-k)} \times M_{(m-k)\times k}\},$$

$$\phi(A, B, C) = \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix}. $$

If $X \in M(m \times n; k)$ is arbitrary, we may find a $C^\infty$ diffeomorphism $h$ of $M_{m \times n}$ (defined by row/column exchanges) such that $h(X) \in U$.

Preliminaries 2: sets of measure zero in a manifold.

$X \subset \mathbb{R}^n$ has measure zero if $\forall \epsilon > 0$ one may find a countable covering $X \subset \bigcup_{i \geq 1} C_i$ by cubes s.t. $\sum_i \text{vol}(C_i) < \epsilon$.

A countable union of sets of measure zero has measure zero.

$\text{meas}(X) = 0$ iff for any $p \in X$ we have a neighborhood $V$ of $p$ in $\mathbb{R}^n$ s.t. $\text{meas}(V \cap X) = 0$.

If $\text{meas}(X) = 0$ and $f \in \text{Lip}_\text{loc}(X; \mathbb{R}^m)$, we have $\text{meas}(f(X)) = 0$.

(Exercise 1.)

¹Adapted from ‘Variedades Diferenciáveis’, by Elon L. Lima, IMPA monograph 1973 (unfortunately untranslated)
Definition: $X \subset M$ has measure zero in $M$ if for any $p \in M$ there exists a chart $(U, \varphi)$ at $p$ s.t. \( \text{meas}(\varphi(U \cap X)) = 0 \) in $\mathbb{R}^n$.

Proposition 1: If $f : M^m \to \mathbb{R}^s$ is $C^1$ and $m < s$, \( \text{meas}(f(M)) = 0 \) in $\mathbb{R}^s$.

Proof. Given $p \in M$, by considering a chart $(U, \varphi)$ at $p$ we may assume there exist cubes $C = C' \times C'' \subset \mathbb{R}^s = \mathbb{R}^m \times \mathbb{R}^{m-s}$ and $\tilde{f} : C \to \mathbb{R}^s$ (Lipschitz) so that $f(U) = \tilde{f}(C' \times \{0\})$.

Proposition 2. Consider the following situation: $f \in C^1(M^m; \mathbb{R}^s)$, $N \subset \mathbb{R}^s$ a (smooth) surface of codimension greater than $m$. Claim: For every $v \in \mathbb{R}^s$ except for a set of measure zero, the translate of $f(M)$ by $v$ misses $N$: \[ [f(M) + v] \cap N = \emptyset. \]
Indeed, define $\phi : M \times N \to \mathbb{R}^s$ by $\phi(p, q) = f(p) - q$. If the intersection is non-empty, $-v$ is in the image $\phi(M \times N)$. But since $\text{dim}(M) + \text{dim}(N) < s$, this image has measure zero in $\mathbb{R}^s$ (by prop. 1).

The same conclusion holds if instead of a single $N$ we have $X = \bigcup_{i \geq 1} N_i$, where each $N_i$ has codimension greater than $m$ in $\mathbb{R}^s$.

3. Local versions of the immersion theorem in $\mathbb{R}^m$.

Proposition 3. Let $f : B^m_3 \to \mathbb{R}^s$ be a $C^r$ map, $r \geq 2$. Assume $s \geq 2m$. Then $\forall \epsilon > 0$ we may find an immersion $g : B^m_3 \to \mathbb{R}^s$ with $|f - g|_{C^1(B_3)} < \epsilon$.

Proof. The set of $s \times m$ matrices of rank $i < m$ is a surface $N_i \subset M_{s \times m}$ of codimension $(m - i)(s - i) \geq 1(2m - (m - 1)) = m + 1$.

Remark: Thus proposition 2 implies the density result for linear transformations: if $T \in M_{s \times n}$ with $s \geq 2m$, $T+A$ has rank $m$ (that is, is a linear immersion $\mathbb{R}^m \to \mathbb{R}^s$) for all $A \in M_{s \times n}$ except for a set of measure zero in $M_{s \times n}$, in particular for $A$’s of arbitrarily small norm.

Let $g(x) = f(x) + Ax$, for $A \in M_{s \times m}$. Then $dg(x) = df(x) + A$, and since $df \in C^1(B_3; M_{s \times m})$, proposition 2 implies $df + A$ has rank $m$ for all $A$’s except for a set of measure zero in $M_{s \times m}$, in particular for $A$’s of arbitrarily small norm.

The perturbation can be localized: there exists $h \in C^r(B_3; \mathbb{R}^s)$ so that $|h-f|_{C^1(B_3)} < \epsilon$ and $h = f$ in $(B_2)^c$. Just let $h(x) = f(x) + \phi(x)(g(x) - f(x))$, with $g$ as in the last paragraph and $\phi \equiv 1$ in $B_1$, $\phi \equiv 0$ off $B_2$.

Where $f$ is already an immersion, it is possible to leave it unchanged (this will be used to globalize the result):
Proposition 4. In the setting of prop.3, assume $F \subset B_3$ is closed and $f|_F$ an immersion. Then we may find $h \in C^r(B_3;\mathbb{R}^s)$ immersion in $\bar{B}_1 \cup F$, with $|h - f|_{C^1(B_3)} < \epsilon$, satisfying $h = f$ in $F \cup (B_2)^c$.

Proof. Let $K = F \cap \bar{B}_2$, which is compact in $B_3$. It is clearly enough to prove the statement replacing $F$ by $K$. Let $V \subset B_3$ be a relatively compact, open neighborhood of $K$, with the property that $f|_V$ is still an immersion. Then:

$$\bar{B}_1 \cup K \subset (\bar{B}_1 - V) \cup \bar{V}.$$ 

Find $\xi$ smooth, supported in $B_3$, so that $\xi \equiv 1$ in $\bar{B}_1 - V$ and $\xi \equiv 0$ in $K \cup (\bar{B}_3 - B_2)$, then set:

$$h = f + \xi(g - f),$$

where $g$ is an immersion close to $f$ in $B_3$ (as in prop. 3), indeed sufficiently close that $|h - f|_{C^1} < \epsilon$ in $B_3$, so that in particular $h$ is still an immersion in $\bar{V}$ (by the ‘simple fact’ below in Exercise 2). Since, in addition, $h = g$ in $\bar{B}_1 - V$, we conclude $h$ is an immersion in $\bar{B}_1 \cup K$. Finally, $h = f$ in $K \cup (B_2)^c$ since $\xi \equiv 0$ on this set.

4. The Whitney and compact-open topologies.

Definitions. We consider two topologies in spaces of $C^r$ $(r \geq 1)$ maps from a manifold $M^m$ to $\mathbb{R}^s$.

1) Whitney $C^1$ topology. Basic neighborhoods of $f \in C^r(M;\mathbb{R}^s)$ are defined by a choice of positive continuous function $\epsilon \in C^0(M;\mathbb{R}_+)$:

$$W^1(f;\epsilon) := \{g \in C^r(M;\mathbb{R}^s); |f(p) - g(p)| < \epsilon(p), |df(p) - dg(p)| < \epsilon(p), \forall p \in M\}.$$ 

(Note: we assume $M$ is endowed with a fixed Riemannian metric.) Although the resulting topology is not metrizable if $M$ is non-compact, we abbreviate the two inequalities in the definition by the notation $|f - g|_{C^1} < \epsilon$ (here $|f - g|_{C^1}$ and $\epsilon$ are functions on $M$).

Remark: An equivalent topology is obtained by fixing a countable, locally finite covering $\mathcal{U} = \{U_i\}_{i \geq 1}$ by relatively compact open sets and considering sequences $a = (a_i)_{i \geq 1}$ of positive reals, taking as the associated basic neighborhood of $f$:

$$W^1(f;a) = \{g \in C^r(M;\mathbb{R}^s); |f - g|_{C^1(U_i)} < a_i \forall i \geq 1\}.$$
2) Compact-open topology. This is the topology of uniform $C_1$ convergence on compact sets. A basic neighborhood of $f$ is defined by choosing $K \subset M$ compact and $\delta > 0$:

$$V^1(f; K, \delta) = \{g \in C^r(M; \mathbb{R}^s); |f - g|_{C_1(K)} < \delta\}.$$ 

This topology has a countable basis and is metrizable. For compact manifolds both topologies coincide, but in the non-compact case the Whitney topology is finer: it has enough open sets that many natural classes of maps (immersions, submersions) are open.

**Proposition 5.** The class of $C^r$ immersions $\text{Imm}^r(M; \mathbb{R}^s)$ is open in $C^r(M; \mathbb{R}^s)$, in the Whitney $C^1$ topology.

**Proof.** Note the following simple fact (Exercise 2): if $f \in C^1(U; \mathbb{R}^s)$ is an immersion (where $U \subset \mathbb{R}^m$) and $K \subset U$ is compact, there exists $\eta > 0$ s.t. if $g \in C^1(U; \mathbb{R}^s)$ and $|g - f|_{C_1(K)} < \eta$, then $g|_K$ is an immersion.

We use the definition of the Whitney topology via bases defined by sequences. Fix a countable, locally finite covering $\{U_i\}_{i \geq 1}$ by relatively compact domains of charts. Let $f \in C^r(M; \mathbb{R}^s)$ be an immersion. From the ‘simple fact’ in the last paragraph, we may find $a_i > 0$ so that if $|f - g|_{C_1(U_i)} < a_i$, then $g|_{U_i}$ is an immersion. Thus the Whitney $C^1$ basic neighborhood $W^1(f; a)$ with $a = (a_i)$ consists of immersions.

**Remark.** Openness of immersions is not true for the compact-open topology (as simple examples show.)

5. The Whitney immersion theorem.

**Theorem.** Assume $s \geq 2m$, $r \geq 2$ The class $\text{Imm}^r(M^m; \mathbb{R}^s)$ of immersions is dense in $C^r(M^m; \mathbb{R}^s)$, endowed with the Whitney $C^1$ topology.

This means: given $f \in C^r(M; \mathbb{R}^s)$ and $\epsilon \in C^0(M; \mathbb{R}_+)\), there exists an immersion $g \in C^r(M; \mathbb{R}^s)$ so that $|g - f|_{C^1} < \epsilon$ on $M$ (as functions on $M$).

Introduce the following global notation: when we consider a locally finite open cover of $M$ by precompact domains of open charts $(U_i, \phi_i)_{i \geq 1}$, we assume for each of these domains $U$ the corresponding chart $\phi$ maps $U = U^{(3)}$ to $B_3 \subset \mathbb{R}^m$, and let $U^{(2)} = \phi^{-1}(B_2)$, $U^{(1)} = \phi^{-1}(B_1)$. We always assume the $U^{(1)}_i$ cover $M$.

**Proof of theorem.** Consider a cover $(U_i)$ as just described. For $n \geq 1$, denote by $M_n \subset M$ the open, precompact submanifold of $M$ consisting of
the union of the first $n \, U_i^{(1)}$’s. Given $f$, assume $f$ has been modified to a $C^r$ map $f_n : M \to \mathbb{R}^s$, so that $f_n$ is an immersion on $M_n$ and $|f - f_n|_{C^1} < \frac{\varepsilon}{2r+1}$ as functions on $U_i^{(1)}$ for $i = 1, \cdots, n$. Let $U^{(3)}$ be the $(n+1)$st domain; we’ll modify $f_n$ to $f_{n+1}$, with $f_{n+1} = f_n$ off $U^{(2)}$. By proposition 4, there exists $h \in C^r(U^{(3)}; \mathbb{R}^s)$ with the properties: (i) $h$ is an immersion in $U^{(1)} \cup (M_n \cap U^{(3)})$ (since $h = f_n$ in $M_n \cap U^{(3)}$); (ii) $h = f_n$ in $U^{(3)} - U^{(2)}$; (iii) $|h - f_n|_{C^1} < \frac{\varepsilon}{2r+1}$, as functions on $U^{(1)}$.

Thus if we define $f_{n+1} \in C^r(M; \mathbb{R}^s)$ by setting:

$$\begin{cases} f_{n+1} = h & \text{on } U^{(3)} \\ f_{n+1} = f_n & \text{on } M - U^{(2)}, \end{cases}$$

we have that $f_{n+1}$ is an immersion on $M_{n+1} = M_n \cup U^{(1)}$, satisfying $|f_{n+1} - f_n|_{C^1} < \frac{\varepsilon}{2r+1}$, as functions on $U^{(1)}$. So we may continue to the next $U^{(3)}$ on the list.

To finish the proof, we set $g(p) = \lim f_n(p)$. This is a pointwise limit, but note that at each $p \in M$ this is a finite sequence (since each $p \in M$ is in only finitely many $U^{(3)}$’s). So in fact the limit is $C^r$-uniform on compact sets, and $g$ is $C^r$, as well as an immersion. It is clear from the construction that $|f - g|_{C^1} < \sum_{n=1}^{\infty} |f_{n+1} - f_n|_{C^1} < \varepsilon$ on $M$. QED

6. Density of injective immersions. First we consider a local stability result in $\mathbb{R}^m$ (similar to the ‘simple fact’ used in the proof of proposition 5).

**Proposition 6.** Let $f \in C^1(U; \mathbb{R}^s)$, where $U \subset \mathbb{R}^m$ is open; let $K \subset U$ be compact and convex; assume $f|_K$ is an embedding. Then there exists $\eta > 0$ s.t. if $g \in C^1(U; \mathbb{R}^s)$ and $|f - g|_{C^1(K)} < \eta$, then $g|_K$ is an embedding (equivalently, an injective immersion).

**Proof.** By prop.5, we have $\eta_1 > 0$ s.t. $|f-g|_{C^1(K)} < \eta_1 \Rightarrow g$ is an immersion on $K$. So we have to work on ‘injective on $K$’.

First, since $f$ is a $C^1$ immersion on the compact set $K$, we may find $\delta > 0$ and $c > 0$ so that $|f(x) - f(y)| \geq c|x - y|$ if $x,y \in K$ and $|x - y| < \delta$. (Exercise 3.) Second, we may always pick $d > 0$ so that $|f(x) - f(y)| \geq d$ if $x,y \in K$ with $|x - y| \geq \delta$.

Let $h = f - g$; note we are assuming $|h|_{C^1(K)} < \eta$ in $K$, and since $K$ is convex the mean-value inequality implies $|h(x) - h(y)| \leq \eta|x - y|$ for $x,y \in K$.

Thus if $x,y \in K$ with $|x - y| < \delta$ we have:

$$|g(x) - g(y)| \geq |f(x) - f(y)| - |h(x) - h(y)| \geq c|x - y| - \eta|x - y|,$$
while if $x, y \in K$ with $|x - y| \geq \delta$:

$$|g(x) - g(y)| \geq |f(x) - f(y)| - |h(x)| - |h(y)| \geq d - 2\eta.$$ 

Thus if we pick $\eta < \min\{\eta_1, c, \frac{d}{2}\}$ we guarantee $g$ is an injective immersion on $K$.

Now for the global version.

**Theorem.** Assume $s \geq 2m + 1$. Then the class of $C^r$ injective immersions is dense in the space of $C^r$ maps from $M^m$ to $\mathbb{R}^s$, with respect to the Whitney $C^1$ topology.

This means: given $f \in C^r(M; \mathbb{R}^s)$ and $\epsilon \in C^0(M; \mathbb{R}^+)$ we may find an injective immersion $g \in C^r(M; \mathbb{R}^s)$ so that $|g - f|_{C^1} < \epsilon$ (as functions on $M$).

**Proof.** We may assume $f$ is an immersion, and that $|g - f|_{C^1} < \epsilon$ implies $g$ is an immersion (and hence, locally an embedding). The issue is injectivity.

The proof is organized as that of the immersion theorem, so refer to global notation in that proof. We pick the $U_i = U_i^{(3)}$ as before, but making sure that if two of them intersect $f$ is injective in their union, also assumed to be the domain of a chart (this is possible since $f$ is locally an embedding). Now let $M_n$ be the union of the first $n$ $U_i^{(1)}$’s, and assume we have changed $f$ to an immersion $f_n \in C^r(M; \mathbb{R}^s)$ which is injective in $M_n$. The goal is to change $f_n$ by a small amount in the $(n + 1)$th. $U_i^{(3)}$ so that the resulting map $f_{n+1}$ coincides with $f_n$ off $U(2)$, and in addition satisfies (with $M_{n+1} := M_n \cup U^{(1)}$):

(i) $f_{n+1}$ is injective in each $U_i^{(3)}$, and if two of the $U_i^{(3)}$’s intersect, $f_{n+1}$ is injective in their union;

(ii) $f_{n+1}$ is injective in $M_{n+1}$.

(iii) $|f_{n+1} - f_n|_{C^1} < \frac{\epsilon}{2n+1}$, as functions on $M$.

The idea is to consider perturbations of $f$ of the following form. Pick $\lambda \in C^k(M; [0, 1])$ (say $M$ is $C^k$, $k \geq r$) so that $\lambda \equiv 1$ in $U^{(1)}$, $\lambda \equiv 0$ off $U^{(2)}$ and also in $M_n - U^{(1)}$. Then choose $0 \neq v \in \mathbb{R}^s$ and define $f_{n+1} \in C^r(M; \mathbb{R}^s)$ by:

$$f_{n+1}(p) = f_n(p) + \lambda(p)v.$$ 

Recall that in proposition 6 we saw that ‘injectivity of immersions is stable on compact subsets of $\mathbb{R}^m$’$^2$, which certainly transfers to chart do-

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$^2$We should be careful about what informal statements like this one are really saying—consider the ‘figure 6 example’
mains on $M$. (i) holds for $f$, and inductively for $f_n$, so (i) (and clearly also (iii)) will also for $f_{n+1}$, provided $v$ is picked with sufficiently small norm. As for (ii), we do know that $f_{n+1}$ is injective in $M_n - \bar{U}^{(3)}$ (since it equals $f_n$ on this set, and $f_n$ is injective on $M_n$). So the issue is ensuring that $f_{n+1}(p) \neq f_{n+1}(q)$ if $p \in M_n - \bar{U}^{(3)}$ and $q \in U^{(1)}$.

So we pick $v \in \mathbb{R}^s$ with small norm, and so that:

$$ [f_n(U^{(3)}) + v] \cap f_n(M_n - \bar{U}^{(3)}) = \emptyset. $$

(Why can we do this? Refer to proposition 2, applied to the $m$-dimensional manifold $U^{(3)}$ and the $m$-dimensional surface $f_n(M_n - \bar{U}^{(3)})$ in $\mathbb{R}^s$, which has codimension $s - m$, strictly greater than $m$ since $s \geq 2m + 1$.)

Then with $p$ and $q$ as above: $f_{n+1}(p) = f_n(p)$ and $f_{n+1}(q) = f_n(q) + v$, so they can’t coincide. We conclude (iii) also holds, and we may proceed.

To finish the proof, let $g(p) = \lim f_n(p)$. This is a pointwise limit, but as noted earlier the sequence is finite at each $p \in M$, so in fact the convergence is $C^r$-uniform on compact sets and the limit $g$ is a $C^r$ immersion, and satisfies:

$$ |g - f|_{C^1} \leq \sum_{n=1}^{\infty} |f_{n+1} - f_n| < \epsilon $$

(as functions). The limit is also injective, since given any two points $p \neq q$ in $M$ we may find $n \geq 1$ so that $g(p) = f_n(p), g(q) = f_n(q)$ and $p, q \in M_n$. QED

7. Whitney’s embedding theorem.

**Theorem.** If $m \geq 2s + 1$ and $M^m$ is a $C^k$ manifold, there exists a proper embedding of class $C^k$ from $M$ to $\mathbb{R}^s$.

**Proof.** First we find a proper map $f \in C^k(M; \mathbb{R}^s)$. Let $\sum \chi_i \equiv 1$ be a $C^k$ partition of unity on $M$. Then $\lambda(p) := \sum_{i \geq 1} i\chi_i(p)$ defines a proper $C^k$ function on $M$, and if we pick $0 \neq v \in \mathbb{R}^s$ arbitrarily and set $f(p) := \lambda(p)v$, $f$ is a $C^k$ proper map. (Verify this as Exercise 4.)

By the previous theorem, we may find an injective immersion $g \in C^k(M; \mathbb{R}^s)$ so that $|f - g|_{C^1} < 1$ on $M$. Hence their limit sets coincide, and are both empty: $L(f) = L(g) = \emptyset$; so $g$ is also proper (and therefore is an embedding.) QED

**Remark 1.** In fact we have the following statement: if $\text{Prop}^r(M; \mathbb{R}^s)$ denotes the space of proper $C^r$ maps, then $\text{Emb}^r(M; \mathbb{R}^s)$ (the space of proper
$$C^r$$ embeddings) is dense in $${\text{Prop}}^r(M; \mathbb{R}^s)$$, with respect to the Whitney $C^1$ topology. Right?

Remark 2. It is not true that $${\text{Emb}}^r(M; \mathbb{R}^s)$$ is dense in $${\text{C}}^r(M; \mathbb{R}^s)$$ (with the Whitney $C^1$ topology), and for a very interesting reason: injective immersions $\mathbb{R} \rightarrow \mathbb{R}^3$ in $\mathbb{R}^3$ may exhibit non-trivial recurrence (that is, $f(x) \in L(f)$ for some $x \in \mathbb{R}$), $C^1$-stably.

In preparation for this example, consider the following exercise:

Exercise: (i) Let $f : M \rightarrow \mathbb{R}^s$ be a $C^1$ injective immersion. Recall the limit set of $f$ is defined (for $M$ non-compact) as:

$$L(f) = \{ X \in \mathbb{R}^s; X = \lim x_k \text{ for some sequence } x_k \rightarrow \infty \}.$$ 

Here $x_k \rightarrow \infty$ for a sequence $(x_k) \subset M$ means $x_k$ has no accumulation point in $M$, equivalently: given $K \subset M$ compact, we can always find $k_0 \geq 1$ so that $x_k \in M - K$ for $k \geq k_0$.

Claim. If $f(M) \cap L(f) \neq \emptyset$, $f$ is not an embedding into $\mathbb{R}^s$.

(ii) Give an example to show that two injective immersions $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$ may satisfy $|f(x) - g(y)| < C$ for all $t$ (and some fixed $C > 0$), but have disjoint limit sets (both non-empty). Is this possible if, say, $L(f) = \emptyset$?

Idea. Consider a curve spiraling onto the unit circle and contained in a vertical strip of width 3, and translate the picture by 3 units. For the second part, find a productive statement equivalent to $L(f) = \emptyset$.

(iii) Say that a continuous positive function $\epsilon \in C^0(M; \mathbb{R}_+)$ satisfies $\epsilon \rightarrow 0$ at infinity if $(\forall \delta > 0) (\exists K \subset M \text{ compact}) (x \in M - K \Rightarrow 0 < \epsilon(x) < \delta)$.

Claim. If $|f - g| < \epsilon$ (as functions on $M$) for a function $\epsilon \in C^0(M, \mathbb{R}_+)$ tending to 0 at infinity, then $L(f) = L(g)$. (Here $f, g \in C^1(M; \mathbb{R}^s)$ are $C^1$ injective immersions.)