

1 (i)  $f: S^2 \rightarrow S^2$  smooth,  $f(x) \neq x \forall x$ .

If  $(1-t)f(x) + t\alpha(x) = 0$  for some  $t \in [0,1]$ , taking norms:

$$(1-t)\|f(x)\| = t\|\alpha(x)\|, \text{ so } t = \frac{1}{2} \text{ (since } \|\alpha(x)\| = \|f(x)\| = 1)$$

and  $f(x) + \alpha(x) = 0$ , or  $f(x) = x$ , contradiction.

Hence  $H(t, x) = \frac{(1-t)f(x) + t\alpha(x)}{\|(1-t)f(x) + t\alpha(x)\|}$  is a homotopy in  $S^2$  from  $f$  to  $\alpha$ .

(iii) Suppose  $V$  is a nonvanishing tangent v.f. on  $S^2$ . Then

$$f(x) = \frac{x + V(x)}{\|x + V(x)\|} \simeq \text{id}_{S^2} \text{ via } H(t, x) = \frac{x + tV(x)}{\|x + tV(x)\|}, x \in S^2, t \in [0,1]$$

$$(\text{note}) \quad \langle x, V(x) \rangle = 0 \text{ implies } \|x + tV(x)\|^2 = 1 + t^2\|V(x)\|^2 \neq 0 \forall t \in [0,1])$$

Since  $\text{id}_{S^2}$  and  $\alpha$  are not homotopic, this shows  $f \neq \alpha$

But  $f(x) = x \Rightarrow 0 = \frac{\|V(x)\|^2}{\|x + V(x)\|}$  (inner prod w/  $V(x)$ ), contradicting  $V(x) \neq 0$ .

Thus  $f$  has no fixed pts, and is not homotopic to  $\alpha$ : contradicts (i).

2 (i) Let  $f(x, y) = (e^x \cos y, e^x \sin y)$

Then  $df(x, y) = e^x \begin{bmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{bmatrix}$  is nonsingular  $\forall (x, y)$ , so

(by the InFT)  $f$  is a local diffeo. But  $f$  is not injective on  $\mathbb{R}^2$

( $f(x, y) = f(x, y+2n\pi) \forall n \in \mathbb{Z}$ ) hence is not a diffeo onto its image.

(Q) Is  $f$  a covering map  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \setminus \{0\}$ ?

(ii)  $f: X \rightarrow Y$  injective local diffeo  $\Rightarrow f$  is a bijective <sup>olif'ble</sup> ~~cont~~ map

w/ diff'ble inverse. So need to show:  $f$  is a homeomorph onto its image; equivalently,

that  $f$  is an open map. Let  $A \subset X^{\text{open}}$   
 $\underset{U \subset A \text{ and }}{\overline{U}} \subset V$  and  $y_0 \in f(A)$ ,  $V \subset Y$  a nbhd of  $y_0$ ,  $U \subset X$   
 $y_0 = f(x_0), x_0 \in A$ .

a nbhd of  $x_0$  st.  $f|_U: U \rightarrow V$  is a diffeo. In part.  $V \subset f(X)$ . So

$f(A)$  is open, and  $f$  is an open map. (Briefly: local diffeos. are open maps)

3 Let  $\{A \mid A \neq 0\} = M^* \subset M_{4 \times 4} \approx \mathbb{R}^4$  (open). Then  $\det: M^* \rightarrow \mathbb{R}$  is a submersion:

$$D(\det)(A)[H] = a_{11}h_{22} + h_{11}a_{22} - a_{21}h_{12} - h_{21}a_{12} \neq 0 \text{ for } H = \begin{cases} a_{ij} \\ \text{if } a_{ij} \neq 0 \text{ (other entries) } \end{cases}$$

so  $\{A \in M^* \mid \det A = 0\}$  is a submanifold of  $M^* \subset \mathbb{R}^4$  of codim 1 (dimension 3). This is exactly the set of rank 1 matrices.

**5** (i) The mod<sub>2</sub> degree is the intersection number  $\Sigma_2(f, \{y\})$ , which is the same for all  $y \in Y$  (since  $Y$  is connected). For  $y \in Y$  a regular value, w/  $f^{-1}(y) = \{x_1, \dots, x_n\}$ , this number is  $n \pmod{2}$ . Since  $\deg_2(f) \neq 0$ ,  $n$  is odd (in part. nonzero). The complement of  $f(X)$  in  $Y$  is open, hence nonempty and regular and all its points are regular values (if  $f(X)^c \neq \emptyset$ ). This is incompatible w/  $\# f^{-1}(y) \neq 0$  for  $y$  a reg. value. Thus  $f(X)^c = \emptyset$ .

(ii)  $f(X)$  is compact, so if  $Y$  is noncompact  $f$  can't be onto, so  $\deg_2(f) = 0$ .

**6**  $\mathbb{R}^3 - L$  deformation retracts to  $S^2 - \{p_1^+, p_1^-, p_2^+, p_2^-, p_3^+, p_3^-\}$  ( $p_i^\pm$  the intersection by radial deformation  $h_t(x) = (1-t)x + t \frac{x}{\|x\|}$ ,  $x \in \mathbb{R}^3 - L$  of  $x_i$  axis w/  $S^2$ )

And  $S^2 - \{p_i^\pm; i=1,2,3\}$  is homeo to  $\mathbb{R}^2 - \{p_1, \dots, p_5\}$  (via stereographic proj.)  $\mathbb{R}^2 - \{p_1, \dots, p_5\}$  def. retracts to a wedge of 5 circles, with  $\pi_1$  equal to  $F_5$  (free gp on 5 gen's).

**4** We may assume  $Y$  is embedded in  $\mathbb{R}^N$ ,

Let  $U_1, \dots, U_N$  be a finite open cover of  $X$  by domains of coord charts  $\varphi_i: U_i \rightarrow \mathbb{R}^m$ , with  $\bar{V}_i \subset U_i$ ,  $\bar{V}_i$  compact and  $V_i$  open, so that  $\{V_i\}_{i=1}^N$  also covers  $X$ . Given  $f, g: X \xrightarrow{C^1} Y \subset \mathbb{R}^N$ , define:

$$\|f - g\|_1 = \sup_{X} |f - g| + \sum_{i=1}^N \|d(f \circ \varphi_i^{-1}) - d(g \circ \varphi_i^{-1})\|_{\sup \text{ norm in } L(\mathbb{R}^m, \mathbb{R}^N)}$$

$$(\varepsilon \in \mathbb{R}_+) \quad V(f, \varepsilon) = \{g: X \xrightarrow{C^1} Y \subset \mathbb{R}^N; \|f - g\|_1 < \varepsilon\}. \quad (\text{basis of nbds of } f \text{ in } C^1(X, Y))$$

Note  $f \circ \varphi_i^{-1} \in C^1(W_i, \mathbb{R}^n)$ ,  $W_i = \varphi_i(U_i) \subset \mathbb{R}^m$  (open)

for  $x \in W_i$ ,  $d(f \circ \varphi_i^{-1})(x) \in L(\mathbb{R}^m, \mathbb{R}^n)$ , and we take the Hilbert-Schmidt norm.

Let  $\mathcal{S} = L(\mathbb{R}^m, \mathbb{R}^N)$  be the open set of operators with rank  $\geq n = \dim Y$ . ( $n < N$ )

Let  $f: X \rightarrow Y$  be a  $C^1$  submersion. Then for each  $x \in U_i$ ,  $d(f \circ \varphi_i^{-1})(x) \in \mathcal{S}$

By continuity  $\text{dist}_L((f \circ \varphi_i^{-1})(\bar{V}_i), \mathcal{S} \setminus \mathcal{D}) = \varepsilon_i > 0$  ( $\varphi(\bar{V}_i) \subset W_i$  compact)

If  $\varepsilon < \min_{1 \leq i \leq N} \varepsilon_i$  and  $g \in V(f, \varepsilon)$ ,  $\text{rank}(d(g \circ \varphi_i^{-1})(x)) \geq n$  for  $x \in \varphi(\bar{V}_i)$ . (i.e.)

(hence equals  $n$ , since  $\dim \text{Ran } d(g \circ \varphi_i^{-1}) \leq n$ ). Thus  $dg(\dot{f})$  is onto  $Y \ni x$ , so  $g$  is a submersion