

1. (i) parametrize S^1 via $\exp : [0, 2\pi] \rightarrow \mathbb{C}$, $\exp(x) = e^{ix}$ (bijection)

$$\text{Then } f^{-1}(\exp(x_0)) = \left\{ \exp\left(\frac{x_0}{2}\right), \exp\left(\frac{x_0}{2} + \pi\right) \right\}$$

If $\delta > 0$ is small so that $x_0 + \delta < 2\pi$ and $x_0 - \delta > 0$, then

f maps $\exp\left(\frac{x_0 - \delta}{2}, \frac{x_0 + \delta}{2}\right)$ and $\exp\left(\frac{x_0 - \delta}{2} + \pi, \frac{x_0 + \delta}{2} + \pi\right)$

homeomorphically onto $(x_0 - \delta, x_0 + \delta)$. f is a local homeo. and S^1 is compact,
(surjective)

so f is a covering. Since $\pi_1(S^1)$ is abelian, the cover is regular.

(ii) Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$ be a representative of a generator $u = [\gamma]_{S^1}$ of $\pi_1(S^1, 1) \approx \mathbb{Z}$. Then $f(\gamma(t)) = e^{2it}$, $t \in [0, 2\pi]$, so $[f \circ \gamma]_{S^1} = 2u$.

Thus $f_* \pi_1(S^1, 1) = \{2nu; n \in \mathbb{Z}\} \approx 2\mathbb{Z}$ (the subgroup of even integers)

$\text{Aut } \approx \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ (acting on S^1 by $z \mapsto -z$, $z \in \mathbb{C}, |z| = 1$).

2. (i) G acts properly discontinuously in Y if $(\forall y \in Y) \exists U \subset Y$ nbhd of y s.t. $gU \cap U = \emptyset \quad \forall g \in G, g \neq \text{id}$.

(ii) Let $\tilde{x} \in \tilde{X}$, $V \subset X$ be an evenly covered nbhd of $x = p(\tilde{x})$.

Then $p^{-1}(V) = \bigsqcup_{\lambda} U_{\lambda}$ (disjoint) and with U_{λ_0} a nbhd of \tilde{x} .

If $g \in \text{Aut}(\tilde{X}|X)$, $g \neq \text{id}$, then $g(U_{\lambda_0}) = U_{\lambda}$ (for some $\lambda \neq \lambda_0$), hence $g(U_{\lambda_0}) \cap U_{\lambda_0} = \emptyset$.

3. Let $r : M \rightarrow S^1 \vee S^1$ be a retraction onto the figure-eight.

Then $i_k : \pi_1(S^1 \vee S^1) \rightarrow \pi_1(M)$ is injective. Indeed $r_* \circ i_k = \text{id}_{F_2}$,

where $F_2(a, b)$ (free group) is $\pi_1(S^1 \vee S^1, x_0)$. Let

$f : S^1 \vee S^1 \rightarrow S^1$ (cont.) map one circle to x_0 , and \cong the identity on the other.

Then $f_*(F_2(a, b)) = \langle b \rangle \approx \pi_1(S^1)$ (say), hence is nontrivial.

Let $g : f \circ r : M \rightarrow S^1$ (cont.). Then $g_* = r_* \circ f_*$.

g_* can't be trivial in π_1 ; if it were, $g_* \circ i_k = f_* \circ (r_* \circ i_k) = r_*$ would be trivial, contradiction. Hence g doesn't lift to R .

4 Let $M \subset \mathbb{R}^n$ be a smooth hypersurface. Then $\mathbb{R}^n \setminus M = D \sqcup V$, with D open bounded, and $M = \partial \bar{D}$. \bar{D} (a manifold with boundary) inherits the orientation from \mathbb{R}^n , and then we may orient M using the outward normal (the normal vector to M pointing to the outside of D ; i.e., into V).

5 (i) $f: (U, 0) \rightarrow (\mathbb{R}^k, 0)$ smooth, $B \subset U$ ball s.t. w/ ctr 0 s.t. \bar{B} contains no other elements of $f^{-1}(0)$

0 is a regular point, so $df(0) \in \mathcal{L}(\mathbb{R}^k)$ is an isomorphism.

$$f(x) = df(0)[x] + r(x) \text{ near } 0, \text{ w/ } \frac{r(x)}{|x|} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Consider $f_t(x) = \begin{cases} df(0)[x] + \frac{r(tx)}{t}, & t \in (0, 1] \\ df(0)[x], & t = 0 \end{cases}$

Then $f_1(x) = f(x)$ and f is cont. int. $t \in [0, 1]$ (uniformly over a cpt ball at 0),

since $\lim_{t \rightarrow 0} f_t(x) = f_0(x)$. Thus $(f_t)_{t \in [0, 1]}$ is a homotopy from f to

the isomorphism $df(0)$. If $\det df(0) > 0$, a further homotopy connects $df(0)$ to the identity (since GL_k^+ is connected). Thus $\deg(\frac{g}{|g|}) = W(g, 0)$
 $= \deg(\frac{f|_{\partial B}}{|f|_{\partial B}}) = \deg \mathbb{I}_k|_{\partial B} = 1$. Composing w/ a reflection gives $W(g, 0) = -1$
if $\det df(0) < 0$.

(ii) Let $f^{-1}(z) = \{x_1, \dots, x_N\} \subset B$. Choose balls B_i w/ ctr x_i so that $\overbrace{\text{disjoint}}$

$W(f|_{\partial B_i}, z) = \pm 1$, depending on whether $df(x)$ preserves or reverses orientation, $i=1, \dots, N$ (possible from part (i)). Considering $\hat{B} = B \setminus \bigsqcup_{i=1}^N B_i$, we see that $W(f|_{\partial \hat{B}}, z) = 0$

since f extends smoothly from \hat{B} to $\mathbb{R}^k \setminus \{z\}$. But $W(f|_{\partial B}, z) = W(f|_{\partial B}, z) - \sum_i W(f|_{\partial B_i}, z)$

Hence $W(f|_{\partial B}, z) = \sum_{i=1}^N \sigma(x_i)$, $\sigma(x_i) = \pm 1$. (due to opposite orient.)

6 On $S^n \subset \mathbb{R}^{n+1}$, consider the v.f. $V(x) = e_{n+1} - \langle x, e_{n+1} \rangle x$, $x \in S^n$. note $V(x) \in T_x S^n$, and vanishes only at $x = \pm e_{n+1}$.

We have $dV(e_{n+1}) = -\text{Id}_{\mathbb{R}^n}$, $dV(-e_{n+1}) = \text{Id}_{\mathbb{R}^n}$: e_{n+1} is a sink, $-e_{n+1}$ a source so $\text{Ind}(V, -e_{n+1}) = 1$, $\text{Ind}(V, e_{n+1}) = (-1)^n$ (degree of antipodal map of S^{n-1})

Thus $\chi_{S^n} = 1 + (-1)^n = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$