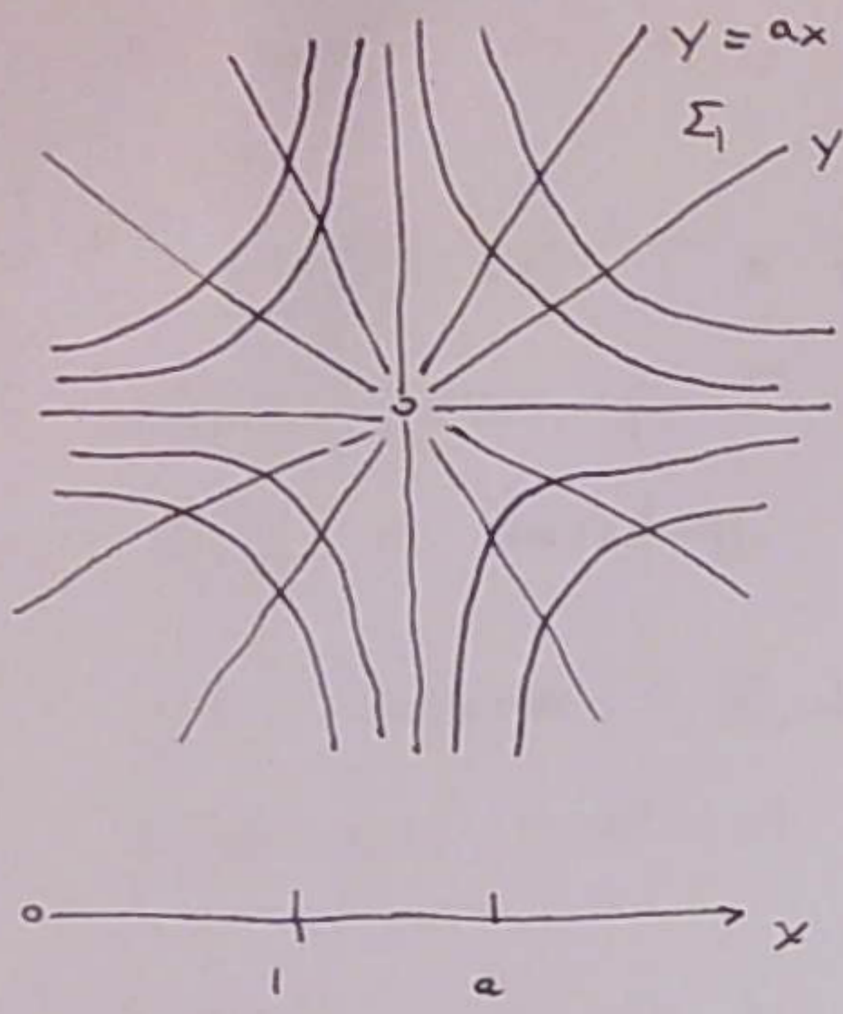
Example The quotient space of a prop. disc. action is not always Hausdorff.

Fix $a>1$, let $\mathbb{Z}$ act on $X = \mathbb{R}^2 \setminus \{0\}$ by $f^n(x,y) = (a^n x, a^{-n} y)$, $n \in \mathbb{Z}$.

The open quadrants and open half-axes are invariant sets. Each open quadrant is foliated by arcs of hyperbola, also invariant



the open interval $(1,a)$ on the $\{x>0\}$ half axis intersects each orbit exactly once. So $X/\mathbb{Z}$ contains a homeomorphic copy of $S^1 = \mathbb{R}_+/\mathbb{Z}$, corresp to $\{x>0\}$

We get a total of 4 circles in $X/\mathbb{Z}$, one for each half-axis.

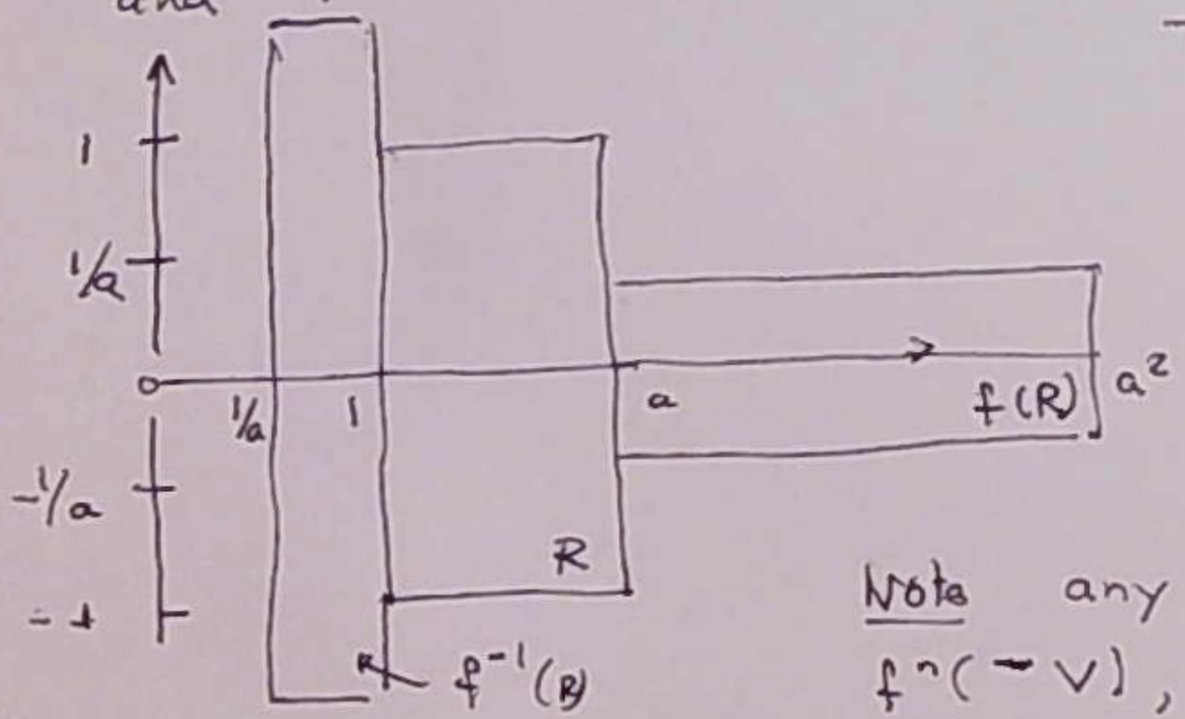
Thus as a set, $X/\mathbb{Z}$ is the disjoint union of 4 cylinders $C_1, \dots, C_4$ and 4 circles $S_1, \dots, S_4$.

In the open 1st quadrant, the open sector Σ_1 (bounded by $\{y=ax\}$ and $\{y=\frac{x}{a}\}$) intersects each orbit exactly once.

Thus $X/\mathbb{Z}$ contains a homeomorphic copy of $(\overline{\Sigma}_1 \setminus \{0\})/\mathbb{Z} \approx S^1 \times \mathbb{R}$, a cylinder C_1.

$\pi : \dot{Q}_1 \longrightarrow C_1$ (onto)

In fact we get 4 disjoint cylinders, w/ $\pi(\dot{Q}_2) = C_2$, $\pi(\dot{Q}_3) = C_3$, $\pi(\dot{Q}_4) = C_4$



The open rectangle $(1,a) \times (-1,1) = \dot{R}$ is mapped disjointly from itself. Thus $p = \pi(t,0) \in S_1$ $(\forall t \in \mathbb{R},\ 1<t<a)$ has a homeomorphic copy of $\dot{R}$ as a nbd in $X/\mathbb{Z}$ (this shows $X/\mathbb{Z}$ is locally euclidean).

Note any nbd of a point in R intersects some $f^n(-V)$, where V is any nbd of a pt. in Σ_1. Thus points in S_1 cannot be separated (by nbds in $X/\mathbb{Z}$) from points in the cylinder C_1 (or from points in C_4).

In the same way, any nbd of a point in Σ_1 intersects $f^n(A)$ for some n, where A is any nbd of a pt in $(1, a) \subset \{x > 0\}$: points in C_1 cannot be separated (by nbds in $X/\mathbb{Z}$) from points in the circle S_1 (or from pts in the circle S_2). The quotient topology in $X/\mathbb{Z}$ is locally euclidean, but not Hausdorff.

Exercise (i) Let $\Gamma = \{(x, gx);\ g \in G,\ x \in X\}$ be the graph of the action. Then if Γ is closed in $X \times X$, the quotient space X/G is Hausdorff. (Here $G \curvearrowright X$ prop. disc., X Hausdorff.)

(ii) Suppose (X, d) is metric, and G acts by isometries: $d(gx, gy) = d(x, y)$ $\forall g, x, y$. Then Γ is closed.

Exercise Let X be a diff'ble manifold (C^r atlas, $r \geq 1$) and G act on X by C^r diffeomorphisms, properly discontinuously and with closed graph. Then

(i) X/G is locally euclidean, Hausdorff, 2nd ctble. (topological mfld)

(ii) X/G admits a C^r atlas.