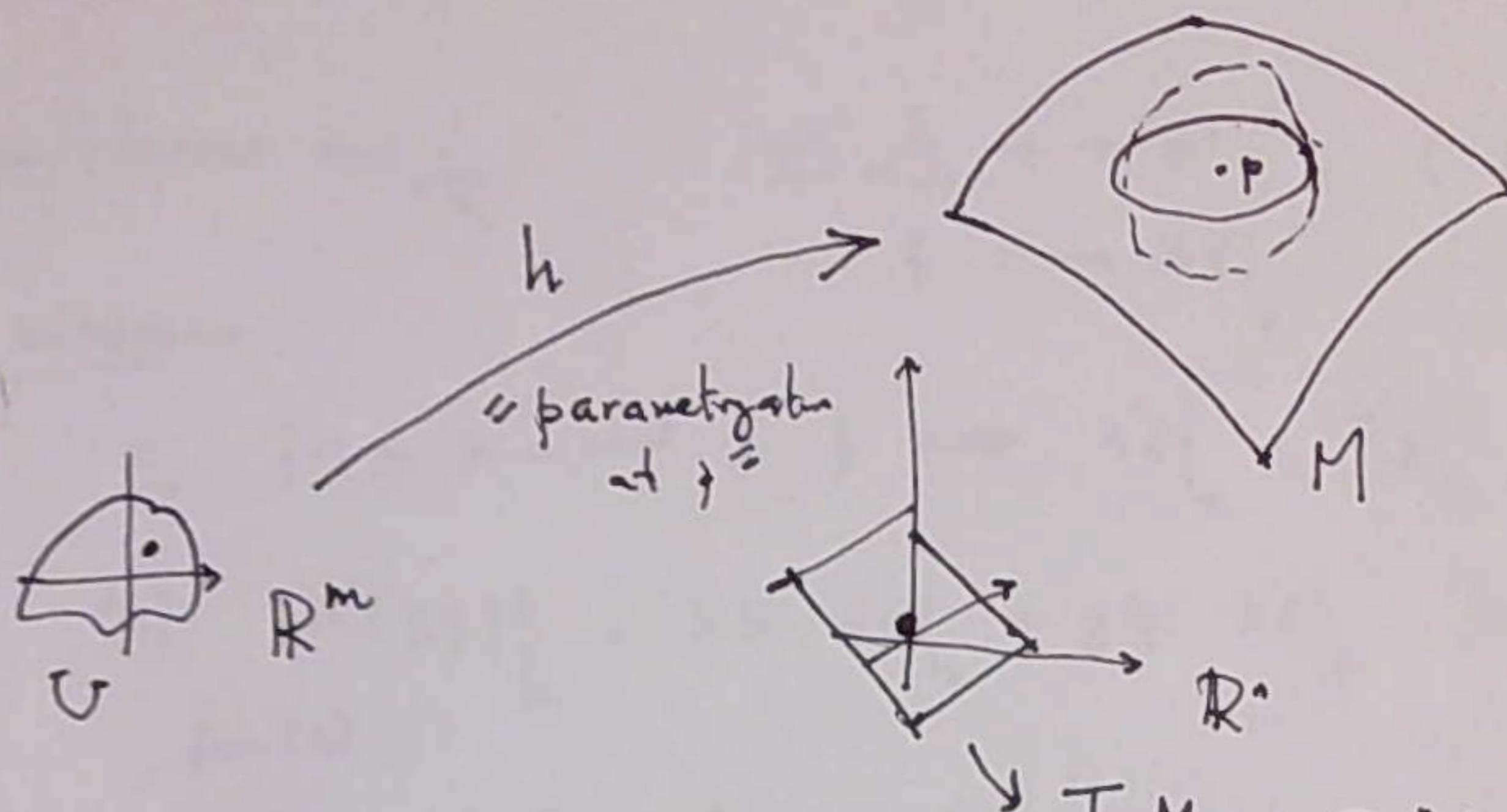


## Surfaces in Euclidean space

y/22 (0)

Def.  $M^m \subset \mathbb{R}^n$   $m$ -dimensional surface of class  $C^k$  ( $k \geq 1$ )  
 $(m \leq n)$



If  $\forall p \in M \exists W \subset \mathbb{R}^n$  open  $\overset{p}{\in} W \subset \mathbb{R}^n$   $T_p M \subset \mathbb{R}^n$  (subspace)

$\exists h: U \rightarrow \mathbb{R}^n$  of class  $C^k$   
 $U \subset \mathbb{R}^m$

s.t.  $h(U) = W \cap M$  and,  $h$  is a homeo onto  $h(U)$

and  $dh(x) \in L(\mathbb{R}^m; \mathbb{R}^n)$  has rank  $m \quad \forall x \in U$ .

tangent space at p Assume  $h(x_0) = p$

$T_p M \stackrel{\text{def}}{=} dh(x_0)[\mathbb{R}^m] \subset \mathbb{R}^n$  ( $m$ -dim'l subspace)

Then (equivalently)

$T_p M = \left\{ \alpha'(0) ; \alpha: I \xrightarrow{C^1} \mathbb{R}^n, \alpha(I) \subset M, \alpha(0) = p \right\}$   
 How do we

Q) Define  $T_p M$  when  $M$  is not a subset of  $\mathbb{R}^n$ ?

Tangent space / tangent bundle / differential

◦ directional derivative at  $p^*$

Def. 1 A tangent vector  $X$  at  $p \in M$ .  $\mathcal{A}_p : \{ \text{smooth fns in a nbd of } p \}$  (or  $C^r$ )

is a linear map  $/ \mathbb{R}$

$$X: \mathcal{A}_p \longrightarrow \mathbb{R}$$

$$f \longmapsto Xf|_p$$

satisfying

$$(1) \quad f = g \text{ in a nbd. of } p \implies Xf|_p = Xg|_p$$

$$(2) \quad X(fg)|_p = f(p) Xg|_p + g(p) Xf|_p \quad (\text{Leibniz})$$

$$(X(\text{const.}))_p = 0, \quad Xf = X(f + \text{const.})$$

$T_p M$  is a vector sp.  $/ \mathbb{R}$

$$(X+Y)f|_p \stackrel{\text{def.}}{=} Xf|_p + Yf|_p \dots$$

basis of  $T_p M$  assoc. to a chart at  $p$ .  $(U, h)$   $h(p) = 0$   
Let  $f \in \mathcal{A}_p$   $(\text{Ass. } f(p) = 0)$

Let  $(x_i)$   $g = \sum f \circ h^{-1} : h(U) \rightarrow \mathbb{R}$  ( $C^r$ , or smooth)

$$g(0) = 0, \quad g \in C^r \text{ so } g(p) = \sum_{i=1}^m x^i g_i(x) \text{ where } g_i(0) = \frac{\partial g}{\partial x^i}(0)$$

so for  $q \in U$ :

$$f(q) = (g \circ h)(q) = \sum_{i=1}^m x^i(q) g_i(h(q))$$

$$(g_i \circ h)(p) = g_i(0) =$$

$$Xf|_p = X(\sum x^i(g_i \circ h)) = \sum_i [(Xx^i)(g_i \circ h)(p) + x^i(p) X(g_i \circ h)]$$

$$= \sum (Xx^i)|_p \partial_i f|_p \quad (\text{where } \partial_i f|_p \stackrel{\text{def.}}{=} \left. \frac{\partial}{\partial x^i} (f \circ h^{-1}) \right|_{x=0})$$

so  $(\partial_i)_{i=1}^m \in T_p M$  is a basis assoc. to the chart  $(U, h)$ .

$$\begin{aligned} \text{Proof} \quad g(x) &= \int_0^1 \frac{dg}{dt}(tx) dt = \int_0^1 \left[ \sum_i x^i \frac{\partial g}{\partial x^i}(tx) \right] dt = \sum_{i=1}^m x^i \underbrace{\int_0^1 \frac{\partial g}{\partial x^i}(tx) dt}_{g_i(x)} \\ &\quad g_i(0) = \frac{\partial g}{\partial x^i}(0) \end{aligned}$$

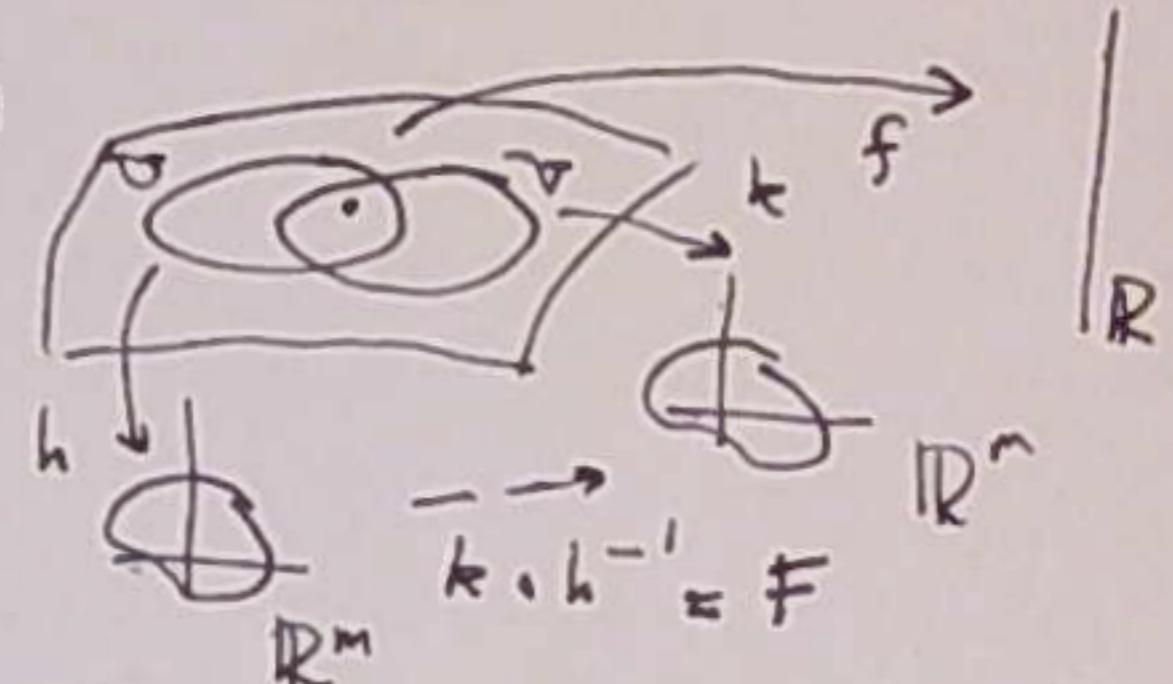
via charts

Def. 2 A tangent vector at  $p \in M$  is a cons. assigning to

each chart  $(U, h)$  at  $\tilde{p}$  a vector  $v \in \mathbb{R}^m$ , so that on overlaps,

if  $(U, h) \mapsto v$  and  $(V, k) \mapsto w$  ( $h(p) = k(p) = 0 \in \mathbb{R}^m$ )

then  $w = d(k \circ h^{-1})(0)[v] \in \mathbb{R}^m$   $\star$



[correspondence between Def. 1 and Def. 2]

Lemma

If  $X \in T_p M$  (def. 1) and

$$\begin{cases} h(q) = (x^i) \in \mathbb{R}^m \\ k(q) = (y^j) \in \mathbb{R}^m \\ q \in U \cap V \end{cases} \quad Xf|_p = \sum_i \cancel{v^i} \partial_{x^i} f|_p \quad (\text{in the basis assoc. to } (U, h))$$

$$= \sum_j w^j \partial_{y^j} f|_p \quad ((V, k))$$

Then for  $F = k \circ h^{-1}$ :

$$w = dF(0)[v].$$

$$v, w \in \mathbb{R}^m$$

Thus a diff. of  $X \in T_p M$  yields an assignment  $(V, k) \mapsto v$  satisfying  $\star$

Conversely given an assignment  $(U, h) \mapsto v \in \mathbb{R}^m$  ( $h(p) = 0$ ) satisfying  $\star$ ,

$$\text{define } Xf|_p = \sum_i v^i \frac{\partial(f \circ h^{-1})}{\partial x^i}|_0.$$

Then this sat (1), (2)  
and is indep. of local chart.  
(exercise)

Differential of a  $C^1$  map

$$f: M \rightarrow N \quad (\text{at } p \in M) \quad df(p) \in L(T_p M; T_{f(p)} N)$$

using (Def. 1) Given  $X \in T_p M$   $df(p)[X] \stackrel{\text{def}}{=} Y \in T_{f(p)} N$

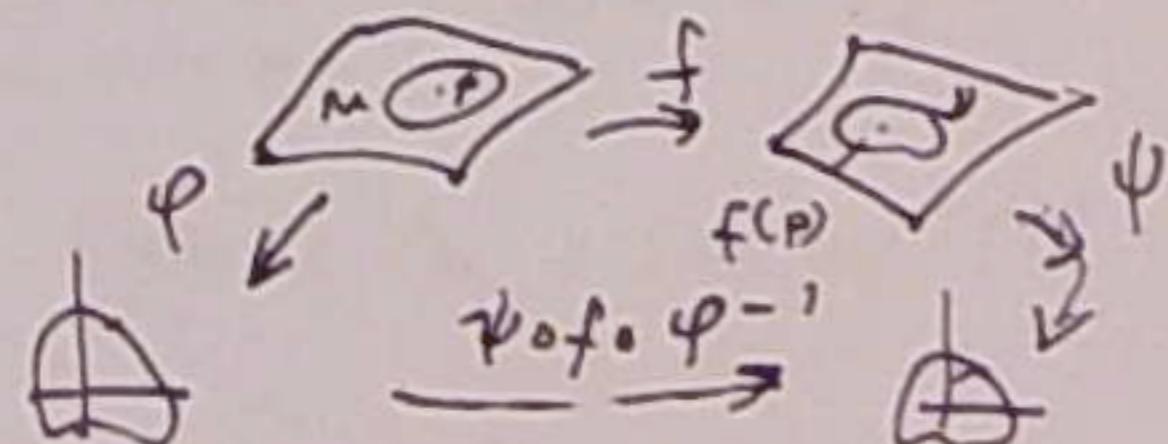
$$\text{where } Y_g|_p = X(g \circ f)|_p \quad \text{for each } g \in \mathcal{D}_{f(p)}^N.$$

using (Def. 2) Let  $X \in T_p M$  be the assignment  $(U, \varphi) \mapsto v$  ( $\varphi(p) = 0 \in \mathbb{R}^m$ )

Then  $df(p)[X]$  is the assignment  $(V, \psi) \mapsto w$  ( $\psi(f(p)) = 0 \in \mathbb{R}^m$ )

where

$$w = d(\psi \circ f \circ \varphi^{-1})(0)[v]$$



$$f \circ h^{-1} = (f \circ k^{-1}) \circ (k \circ h^{-1}) \quad (2.5)$$

$$= (f \circ k^{-1}) \circ F$$

Pf of lemma on 1.2

$$\begin{aligned} X_f &= \sum_i v^i \partial_{x^i} f \Big|_p = \sum_i v^i \frac{\partial}{\partial x^i} (f \circ h^{-1}) = \sum_j w^j \frac{\partial}{\partial y^j} (f \circ k^{-1}) \\ &= \sum_i v^i \frac{\partial}{\partial x^i} [(f \circ k^{-1}) \circ F] \\ &= \sum_{i,j} v^i \left[ \frac{\partial}{\partial y^j} (f \circ k^{-1}) \Big|_{F(x)} \frac{\partial F^j}{\partial x^i} \right] \\ &= \sum_j \frac{\partial}{\partial x^j} (f \circ k^{-1}) \Big|_{F(x)} \left( \sum_i v^i \frac{\partial F^j}{\partial x^i} \right) \quad \forall f \\ \Rightarrow w^j &= \sum_i v^i \frac{\partial F^j}{\partial x^i} \\ \text{i.e. } w &= dF(p)[v] \end{aligned}$$

Exercise (on the def'n of differential)

(i) Prove that  $Y = df(p)[X]$ ,  $Yg \Big|_p = X(g \circ f) \Big|_{f(p)}$  satisfies the conditions in Def. 1 of an element of  $T_{f(p)}N$ .

(ii) Prove that the assignment  $(v, \psi) \mapsto w$  (of vectors in  $\mathbb{R}^n$  assigned to a local chart of  $N$ ) in Def. 2 of  $df(p)[X]$  satisfies the conditions in Def. 2 of an element of  $T_{f(p)}N$ .

Remark A 3rd definition of  $T_p M$  (based on velocity vectors of curves)

Let  $C_p = \{ \gamma : I \rightarrow M \text{ diff'ble}, 0 \in I \subset \mathbb{R}, \gamma(0) = p \}$

Introduce  $\sim$  in  $C_p$  (equiv. rel'n)

$\gamma \sim \mu \Leftrightarrow \exists (\psi, \varphi)$  chart at  $p$  s.t.  $(\varphi \circ \gamma)'(0) = (\varphi \circ \mu)'(0)$ .

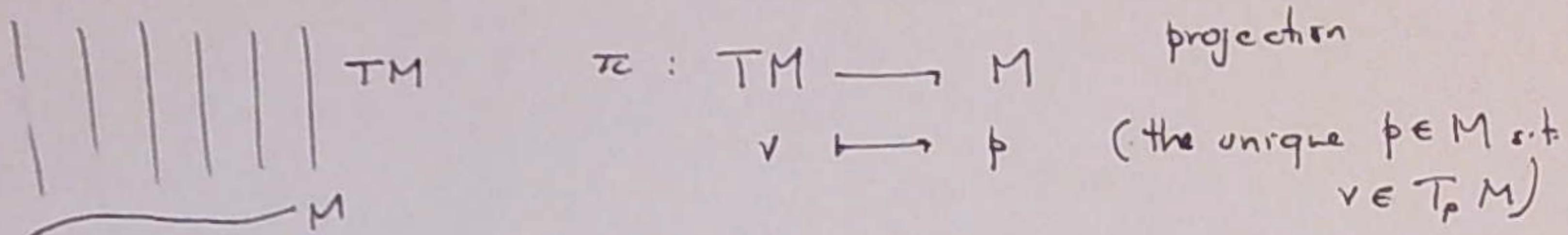
Then set  $T_p M = C_p / \sim$

# tangent bundle to a $C^r$ manifold $M$

③  
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$$TM = \bigcup_{p \in M} T_p M \quad (\text{disjoint union}) \quad [\text{no vector space structure!}]$$

Points  $v \in TM$  are thought of as "vectors w/ basepoint"



## topology of $TM$

(1) require  $\pi^{-1}(U)$  to be open in  $TM$ , if  $U \subset M$  open ( $\pi$  is <sup>i.e.</sup> cont.)

(2) Let  $(U, \varphi)$  be a chart for  $M$ ,  $U \subset M$  open,  $\varphi : U \longrightarrow \mathbb{R}^m$   
 Define  $\tilde{\varphi} : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^m$  via

$$\tilde{\varphi}(v) = (\varphi(p), \alpha) \quad \text{where } p = \pi(v)$$

and  $\alpha \in \mathbb{R}^m$  is assoc.  $p$  to  $(U, \varphi)$  and  $v$

$$\text{or } X_f = \left\{ \sum_i \alpha^i \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \Big|_{x_0}, \varphi(p) = x_0 \right\} \quad (\text{def. 2 see Def. 2})$$

Require  $\tilde{\varphi}$  to be a homeomorphism (def. 2 see Def. 1).

Exercise If charts  $(U, \varphi)$  and  $(V, \psi)$  for  $M$  overlap ( $U \cap V \neq \emptyset$ )

Then  $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^m \longrightarrow \psi(U \cap V) \times \mathbb{R}^m$  is a homeo.

The topology on  $TM$  is well-defined (basis?), Hausdorff and 2nd countable.

$C^{r-1}$  structure on  $TM$   $\tilde{\psi} \circ \tilde{\varphi}^{-1} : (x, \alpha) \mapsto (y, \beta) \quad \alpha, \beta \in \mathbb{R}^m$

where if  $y = F(x)$  is the map  $F = \psi \circ \varphi^{-1}$ , then  $\beta = dF(x)[\alpha]$ .

$F : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$  is a  $C^r$  diffeo.

so  $\tilde{F} : \varphi(U \cap V) \times \mathbb{R}^m \longrightarrow \psi(U \cap V) \times \mathbb{R}^m$

$(x, \alpha) \mapsto (F(x), dF(x)[\alpha])$  is a  $C^{r-1}$  diffeomorphism

(check: )

inverse:  $\tilde{F}^{-1}(y, \beta) = (F^{-1}(y), dF^{-1}(y)[\beta])$ .