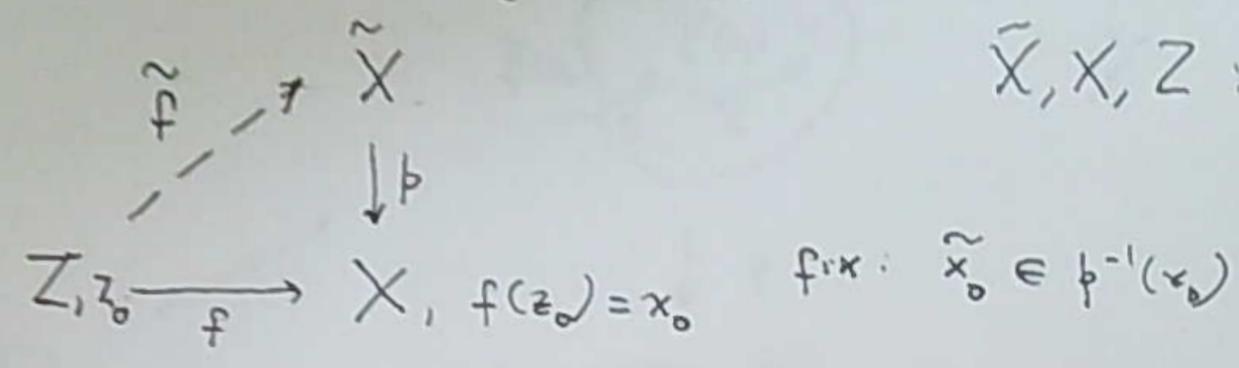


Fundamental lifting theorem



want $\tilde{f} : (Z, z_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ cont. s.t. $p \circ \tilde{f} = f$

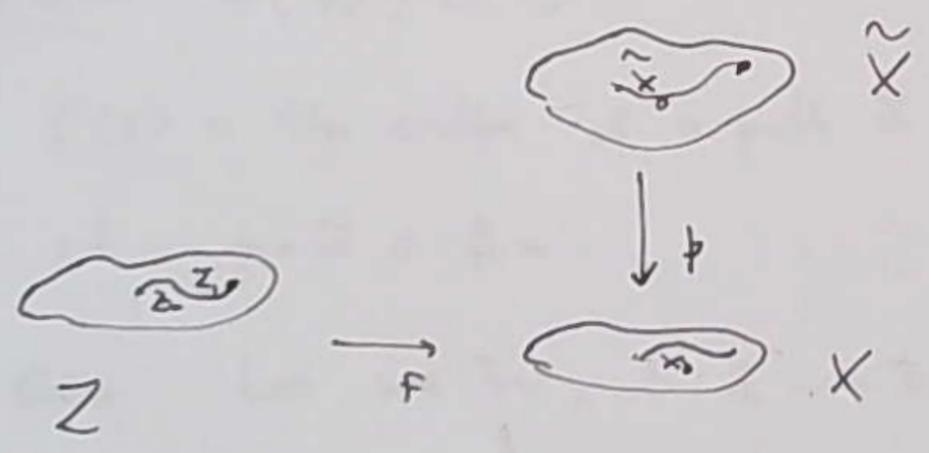
nec. condition \tilde{f}_* , f_* induced hom on π_1

$$f = p \circ \tilde{f} \implies f_* (\pi_1(Z, z_0)) = p_* (\tilde{f}_* \pi_1(\tilde{Z}, \tilde{z}_0)) \subset p_* (\pi_1(\tilde{X}, \tilde{x}_0))$$

$$f_* (\pi_1(Z)) \subset H(\tilde{x}_0) \quad \parallel \quad H(\tilde{x}_0)$$

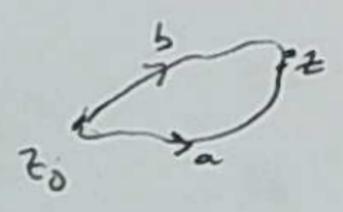
Thm If this holds, the lift \tilde{f} exists (and is unique, given $\tilde{f}(z_0) = \tilde{x}_0$)

Pf.



Given $z_0 \in Z$, let $a: I \rightarrow Z$ be a path from z_0 to z_1 .
Let \tilde{a} be the lift of $f \circ a: I \rightarrow X$ from \tilde{z}_0
set $\tilde{f}(z_1) = \tilde{a}(1)$

① \tilde{f} is well-def. Let $b: I \rightarrow Z$ be a second path from z_0 to z_1

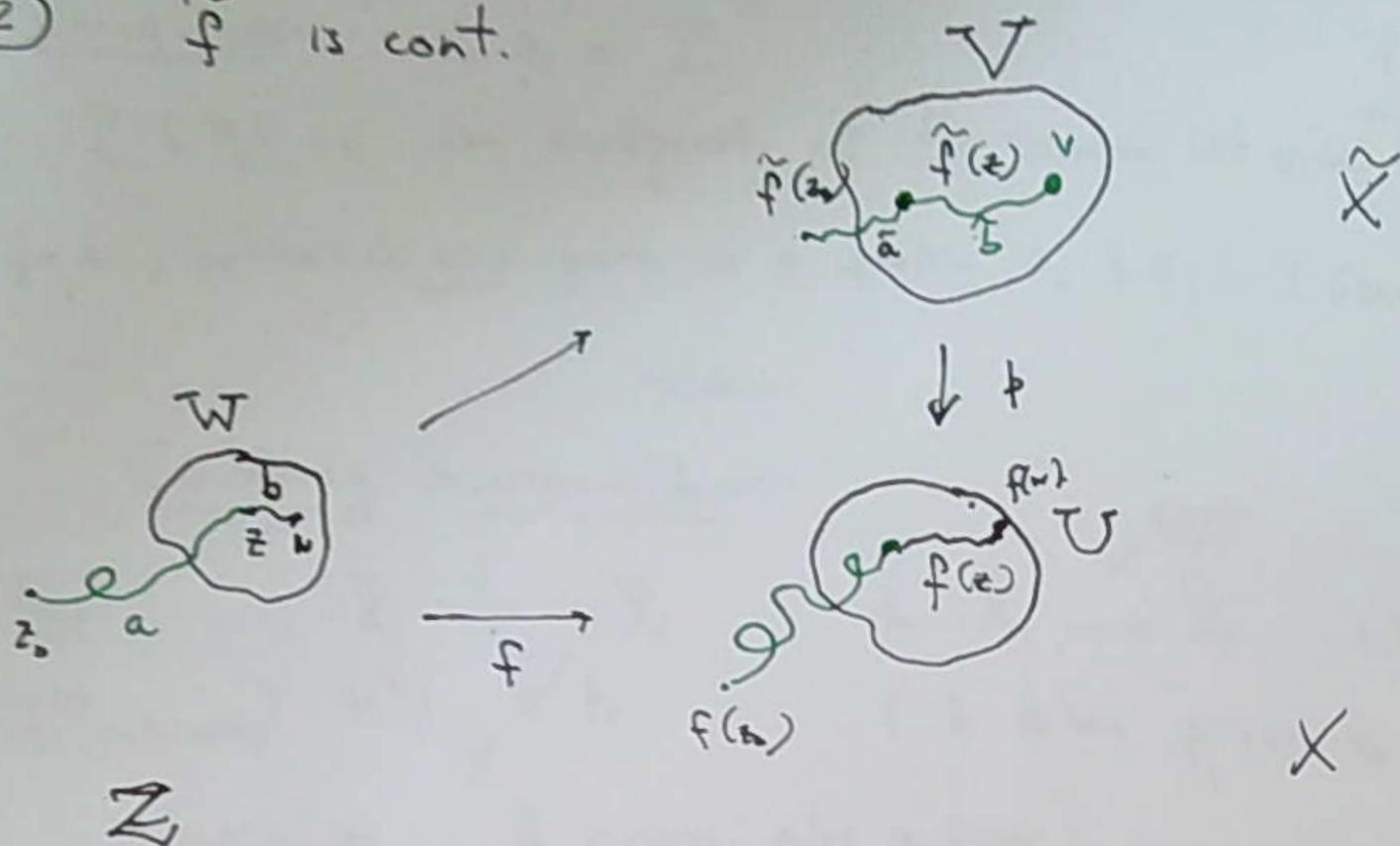


$$b * \bar{a} \in \Omega_{z_0}(Z) \quad f \circ (b * \bar{a}) = (f \circ b) + (f \circ \bar{a}) \in \Omega_{x_0}(X)$$

$$f_* [b * \bar{a}]_Z \in H(\tilde{x}_0) \quad (\text{hypothesis})$$

so $(f \circ b) + (f \circ \bar{a}) \xrightarrow{f_*} \tilde{f} \circ b$ and $\tilde{f} \circ \bar{a}$ have same endpoints. ($\tilde{a}(1) = \tilde{f}(z_1)$)

(2) \tilde{f} is cont.



Let V be a nbd of $\tilde{f}(z)$ in \tilde{X} s.t. $p|_V$ is a homeo onto $U \subset X$ (open nbd of $f(z)$).

f cont. $\Rightarrow \exists W = Z$ (nbd of z), path-connected s.t. $f(W) \subset U$

Claim $\tilde{f}(W) \subset V$.

$\tilde{f}(z)$ is the endpt of a path $\tilde{\alpha}$ in \tilde{X} from $\tilde{x}_0 = \tilde{f}(z)$, s.t. $p \circ \tilde{\alpha} = f \circ a$.

Solve Let $w \in W$, $b: I \rightarrow W$ path in W from z to w

since $p: V \rightarrow U$ is homeo, $\exists \tilde{\beta}$ path in V starting at $\tilde{f}(z)$ (to $v \in V$) s.t. $p \circ \tilde{\beta} = f \circ b$.

Then $\tilde{\alpha} * \tilde{\beta}$ is a path in \tilde{X} , starting at \tilde{x}_0 s.t.

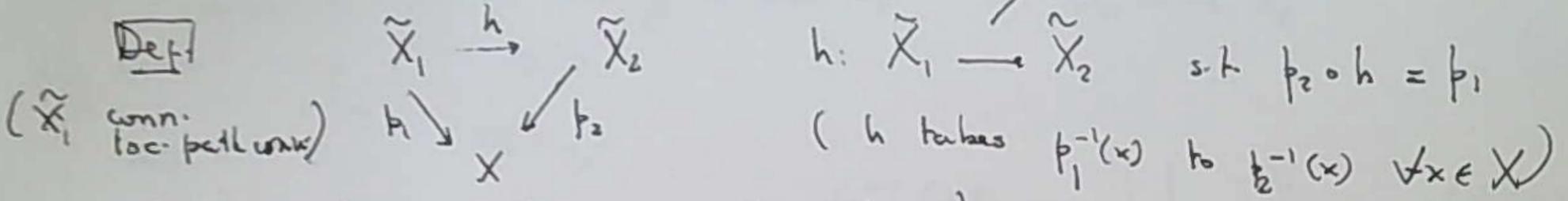
$p \circ (\tilde{\alpha} * \tilde{\beta}) = (p \circ \tilde{\alpha}) * (p \circ \tilde{\beta}) = (f \circ a) * (f \circ b) = f \circ (a * b)$.
so $\tilde{\alpha} * \tilde{\beta}$ lifts $f \circ (a * b)$ over p (unique lift from \tilde{x}_0).

so $(\tilde{\alpha} * \tilde{\beta})(1) = \tilde{f}(w) = v \in V$. Thus $\tilde{f}(W) \subset V$.

Uniqueness if $z_1 \in Z$ from \tilde{z}_0

$\tilde{f}(z_1)$ is the endpoint of the unique lift over p of the path $f \circ \alpha$, where α is any path in Z from z_0 to z_1 . (Since \tilde{f} lifts f over p .)

Covering homomorphism



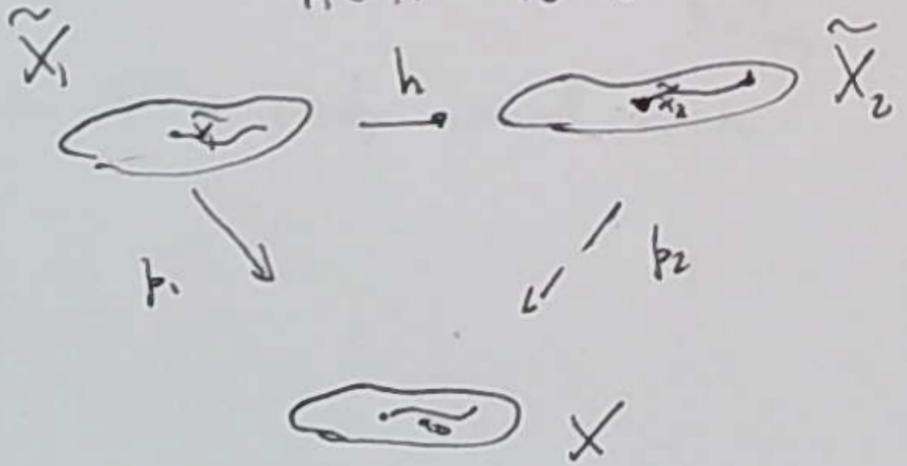
(isomorphism : \exists inverse, also a hom.)
automorphism $h: \tilde{X}_1 \rightarrow \tilde{X}_1$ homeo $p_1 \circ h = p_1$
 $\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{h} & \tilde{X}_1 \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{\text{id}} & X \end{array}$

Prp. if a hom. h exists, then it is uniquely def. by its value at one point.
 (h is a lift of p_1 over p_2)

Prop. Any hom. $h: \tilde{X}_1 \rightarrow \tilde{X}_2$ is a covering map (in part. surjective)

Prf. ① h is surjective

Let $\tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 = h(\tilde{x}_1)$ path from \tilde{x}_2 .
 $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x_0$



Let $a_0 = p_2 \circ a: I \rightarrow X$

Lift a_0 over p_1 to a path

$\tilde{a}: I \rightarrow \tilde{X}_1$ from $\tilde{x}_1 \in p_1^{-1}(x_0)$

Then $h \circ \tilde{a}$ is a lift of a_0

(rel to p_2) from \tilde{x}_2 :

$p_2(h \circ \tilde{a}) = (p_2 \circ h) \circ \tilde{a} = p_1 \circ \tilde{a} = a_0$

Thus $h \circ \tilde{a} = a$

$h(\tilde{a}(1)) = (h \circ \tilde{a})(1) = a(1)$ (arb. pt. in \tilde{X}_2)

So h is ONTO

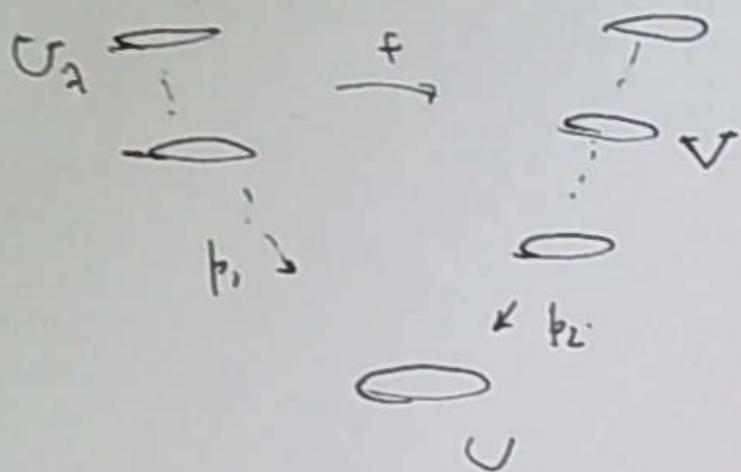
② h is a covering map.

④

Let $\tilde{x}_2 \in \tilde{X}_2$, $x_0 = p_2(\tilde{x}_2)$, $U \subset X$ connected nbd of x_0 in X , evenly covered by both p_1 and p_2 .

Let V be the conn.-component of $p_2^{-1}(U)$ containing \tilde{x}_2 .

Claim V is evenly covered by $h: \tilde{X}_1 \rightarrow \tilde{X}_2$



$$p_1^{-1}(U) = \coprod_{\lambda \in L} \tilde{U}_\lambda \quad (\text{disjoint union})$$

$$p_1: \tilde{U}_\lambda \rightarrow U \text{ homeo.}$$

$$h \upharpoonright (\tilde{U}_\lambda) = p_2^{-1}(U)$$

$h \upharpoonright (\tilde{U}_\lambda)$ is connected and V is a conn. comp. of $p_2^{-1}(U)$

so if $h \upharpoonright (\tilde{U}_\lambda) \cap V \neq \emptyset$ (for some λ) then $h \upharpoonright (\tilde{U}_\lambda) \subset V$

↳ always true for some λ , since f is onto

note: $h \upharpoonright \tilde{U}_\lambda = (p_2 \upharpoonright V)^{-1} \circ (p_1 \upharpoonright \tilde{U}_\lambda)$ is a homeo onto its

composition of homeos.

$$\text{image } (p_2 \upharpoonright V)^{-1} \circ p_1(\tilde{U}_\lambda) = V.$$

$$\underline{\underline{\text{So}}}: h \upharpoonright^{-1}(V) = \coprod_{\lambda \in L_0} \tilde{U}_\lambda$$

$$L_0 = \left\{ \lambda \mid h \upharpoonright (\tilde{U}_\lambda) \cap V \neq \emptyset \right\} \quad (L_0 \neq \emptyset)$$

so V is evenly covered by h .