

(4)

Conjugacy class of a covering

p) $\tilde{X} \rightarrow X$ covering (X, \tilde{X} path-conn.)

[Prop] $\forall x_0 \in X, \tilde{x}_0 \in \tilde{\pi}^{-1}(x_0), \tilde{p}_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.

(follows from unique lifting of homotopies)

Let $H(\tilde{x}_0) \subset \pi_1(X, x_0)$ be the image of \tilde{p}_* (a subgroup of $\pi_1(X, x_0)$).

Q) What does $H(\tilde{x}_0)$ measure?

[Prop] Let $a, b: I \rightarrow X$ from x to y .
 $\tilde{a}, \tilde{b}: I \rightarrow \tilde{X}$ lifts from $\tilde{x} \in \tilde{p}^{-1}(x)$

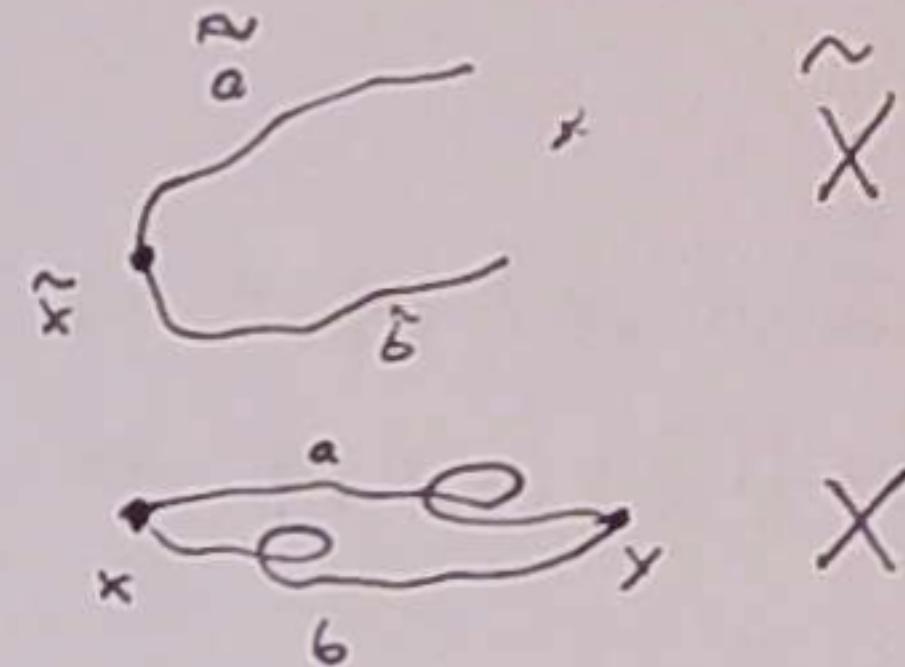
Then $\tilde{a}(1) = \tilde{b}(1) \Leftrightarrow [a * \bar{b}]_x \in H(\tilde{x})$

(\Rightarrow) clear

(\Leftarrow) If $[a * \bar{b}]_x \in H(\tilde{x})$, the lift \tilde{c} of $a * \bar{b}$ from \tilde{x} is closed.

Let $\tilde{a}(s) = \tilde{c}(s/2), s \in [0,1]. \quad (\tilde{p} \circ \tilde{a})(s) = \underbrace{(\tilde{p} \circ \tilde{c})(s/2)}_{a * \bar{b}} = a(s).$
 $\tilde{b}(s) = \tilde{c}(1-s/2)$

Both start at \tilde{x} , end at $\tilde{c}(1/2)$ and lift a, b (resp)



Corollary 1

Let $a \in \Omega_x(X), \tilde{a}$: lift from $\tilde{x} \in \tilde{p}^{-1}(x)$

\tilde{a} is closed $\Leftrightarrow [a]_x \in H(\tilde{x})$

Corollary 2 If \tilde{X} is simply-connected:

(a : loop in X) $a \simeq \text{const.} \Leftrightarrow$ all lifts \tilde{a} are closed

a, b : paths in X $a \simeq b$ (path) \Leftrightarrow all lifts \tilde{a}, \tilde{b} w/same starting pt
 (same endpt) have same endpt.

(2)

The fundamental lifting theorem.

$\tilde{p}: \tilde{X} \rightarrow X$ covering (\tilde{X}, X path-conn)

Z connected, loc. path connected

$f: (Z, z_0) \rightarrow (X, x_0)$ cont.

$$\begin{array}{ccc} & \tilde{f} & \downarrow \\ Z & \xrightarrow{f} & X \end{array}$$

$\boxed{p \circ \tilde{f} = f}$

Thm. Given $\tilde{x}_0 \in p^{-1}(x_0)$, a lift $\tilde{f}: (Z, z_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ exists

iff. $f_*[\pi_1(Z, z_0)] \subset H(\tilde{x}_0)$

Pf. The cond. is necessary, since $f_* \pi_1(Z) = p_*(f_* \pi_1(\tilde{Z}))$

Suppose $f_* \pi_1(Z, z_0) \subset H(\tilde{x}_0)$

$$= p_*(\pi_1(\tilde{X})) = H(\tilde{x}_0)$$

Define $\tilde{f}: Z \rightarrow \tilde{X}$ by: (i) $\tilde{f}(z_0) = \tilde{x}_0$

(ii) given $z \in Z$, let $a: I \rightarrow Z$ be a path from z_0 to z ,

\tilde{a} the lift of $f \circ a: I \rightarrow X$ from \tilde{x}_0 . Set $\tilde{f}(z) = \tilde{a}(1)$
(clearly $p \circ \tilde{f} = f$).

① \tilde{f} is well-def: let $b: I \rightarrow Z$ be a second path from z_0 to z .

Then $b \circ \bar{a} \in \Omega_{z_0}(z)$, so

$(f \circ b) \circ (f \circ \bar{a}) = f \circ (b \circ \bar{a}) \in \Omega_{x_0}(X)$, whose htpy class
 $f_* [\tilde{a}]_Z \in H(\tilde{x}_0)$, by hypothesis.

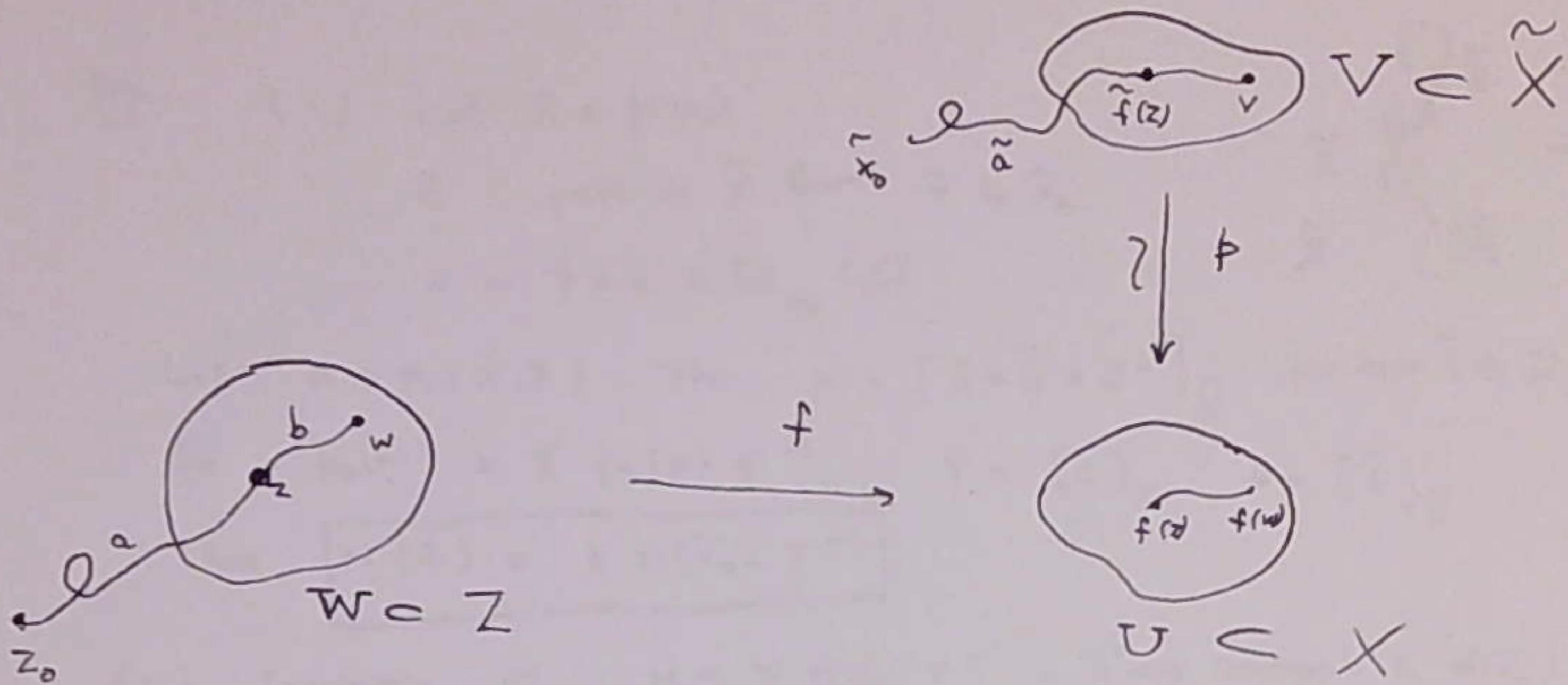
Thus $\tilde{f} \circ b$ and $\tilde{f} \circ \bar{a}$ have the same endpoint.

② \tilde{f} is cont.

Let V be a nbd. of $\tilde{f}(z)$ in \tilde{X} , s.t. $p|_V$ is a homeo onto $U \subset X$,
nbd. of $f(z)$ in X . Let $W \subset Z$ be a path-conn. nbd. of z s.t.
 $f(W) \subset V$. We claim $\tilde{f}(W) \subset V$

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$\tilde{f}(z)$ is the endpoint of a path \tilde{a} in \tilde{X} from \tilde{x}_0 ,
 s.t. $p \circ \tilde{a} = f \circ a$ (a is a path in Z from z_0 to z)



Since $p: V \rightarrow U$ is homeo, $\exists \tilde{b}$ path in V from $\tilde{f}(z)$

to $v \in V$, s.t. $p \circ \tilde{b} = f \circ b$

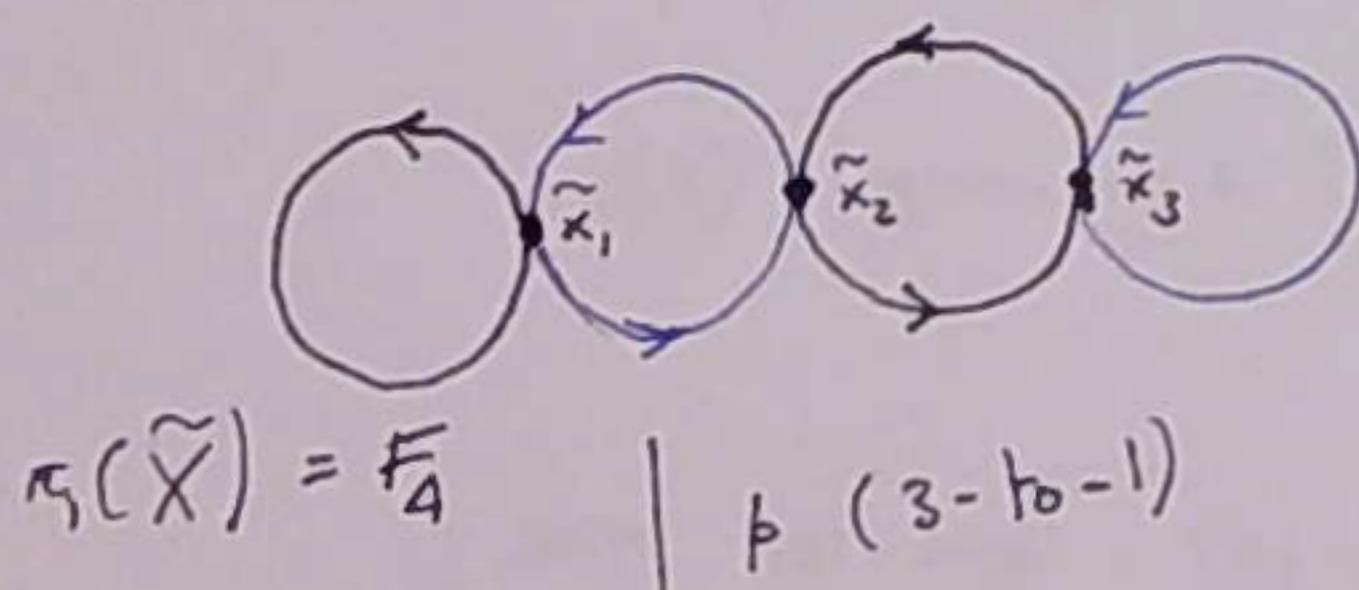
Then $\tilde{a} + \tilde{b}$ is a path in \tilde{X} from \tilde{x}_0 , s.t.

$$p \circ (\tilde{a} + \tilde{b}) = (p \circ \tilde{a}) * (p \circ \tilde{b}) = (f \circ a) * (f \circ b) = f * \circ (a + b)$$

Since $a + b$ joins z_0 to w in Z , by def. $\tilde{f}(w) = (\tilde{a} + \tilde{b})(1) = v$, $\therefore \tilde{f}(w) \in V$.

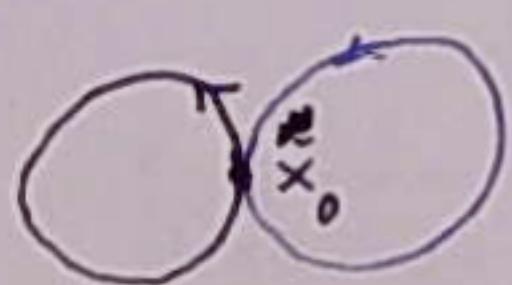
Example

not regular

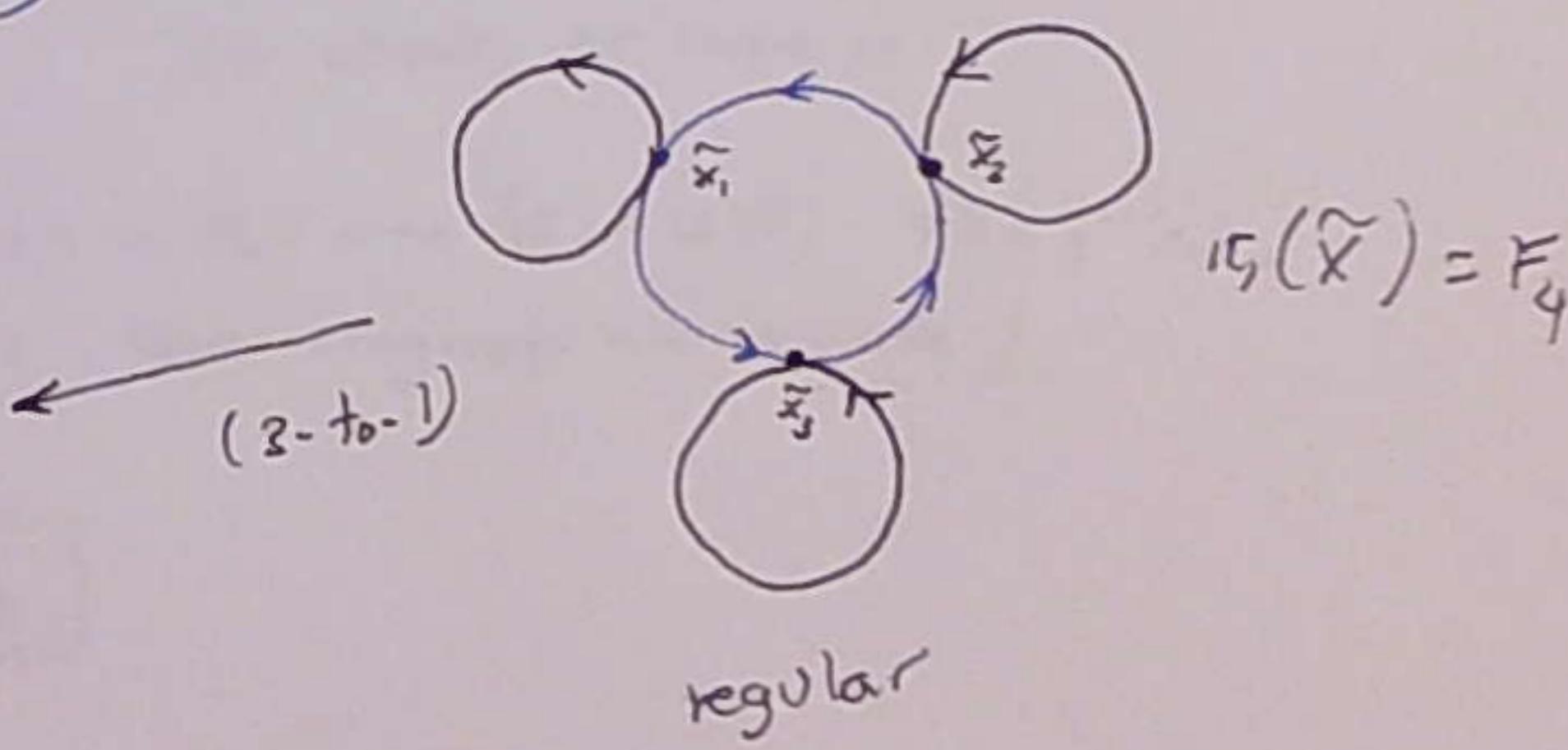


$$\pi_1(\tilde{X}) = F_3$$

$$p \quad (3-to-1)$$



$$\pi_1(X) = F_2$$



$$\pi_1(\tilde{X}) = F_4$$

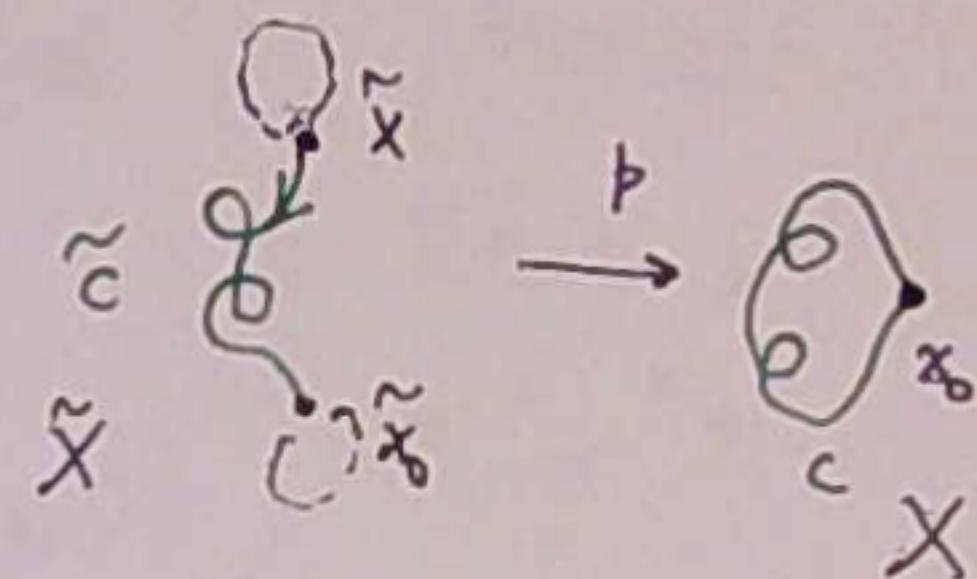
regular

(b)

Prop. As \tilde{x} ranges over $\phi^{-1}(x_0)$, $H(\tilde{x})$ ranges over the whole conjugacy class of $H(\tilde{x}_0)$ in $\pi_1(X, x_0)$.

Pf.(i) Let $\tilde{x} \in \phi^{-1}(x_0)$ \tilde{c} : path in \tilde{X} from \tilde{x} to \tilde{x}_0

$c = \phi \circ \tilde{c} \in \Omega_{x_0}(X)$

Let $\alpha \in \pi_1(\tilde{X}, \tilde{x})$. Then $\alpha = [\tilde{c} * \tilde{b} * \tilde{c}^{-1}]_{\tilde{X}}$ for some $\tilde{b} \in \Omega_{\tilde{x}_0}(\tilde{X})$.

So $\phi_*(\alpha) = \gamma \phi_*(\beta) \gamma^{-1}$, $\gamma = [c]_X$, $\beta = [\tilde{b}]_{\tilde{X}}$

Thus
$$H(\tilde{x}) = \gamma H(\tilde{x}_0) \gamma^{-1}$$

(ii) Conversely, let $H = \gamma H(\tilde{x}_0) \gamma^{-1}$ a group conjugate to $H(\tilde{x}_0)$ in $\pi_1(X, x_0)$.Write $\gamma = [c]_X$, Lift \tilde{c} from \tilde{x}_0 to get $\tilde{\tilde{c}}$, w/ endpoint $\tilde{x} \in \phi^{-1}(x_0)$.Then \tilde{c} starts at \tilde{x} and ends at \tilde{x}_0 , lifting c . Thus $H(\tilde{x}) = \gamma H(\tilde{x}_0) \gamma^{-1}$,so $H = H(\tilde{x})$. □The following are eq. (\tilde{X} path-conn) Fix $x_0 \in X$ (1) For some $\tilde{x}_0 \in \phi^{-1}(x_0)$, $H(\tilde{x}_0) \subset \pi_1(X, x_0)$ is normal(2) The subgroups $H(\tilde{x})$ of $\pi_1(X, x_0)$, $\tilde{x} \in \phi^{-1}(x_0)$, are all normal (and coincide).(3) given $\alpha \in \Omega_{x_0}(X)$, either all its lifts from points $\tilde{x} \in \phi^{-1}(x_0)$ are closed, or none is.(since (3) says $[a] \in H(\tilde{x}_0) \leftrightarrow [a] \in H(\tilde{x}) \quad \forall \tilde{x} \in \phi^{-1}(x_0)$, i.e. these subgroups all coincide).Def. Regular covering.

Def "Covering spaces are classified by a conjugacy class of subgroups of π_1 " (3)

covering $p: \tilde{X} \rightarrow X$ $p': X' \rightarrow X$ $\boxed{\text{equiv.}}$ if $\exists h: \tilde{X} \rightarrow X'$ homeo commuting w/ id_B .

Thm \exists equivalence $h: \tilde{X} \rightarrow X'$, $h(\tilde{x}_0) = h(x'_0)$ iff $\underline{p = p' \circ h}$

$H(\tilde{x}_0) = H(x'_0)$ (subgroups of $\pi_1(X, x_0)$).

$$\tilde{X} \xrightarrow{h} X'$$

$$p \searrow \quad \swarrow p'$$

Pf necessary $H(\tilde{x}_0) = p_* \pi_1(\tilde{X}, \tilde{x}_0) = p'_*(h_* \pi_1(\tilde{X}, \tilde{x}_0))$

$$= p'_*(\pi_1(X', x'_0)) = H(x'_0).$$

sufficiency Ass. $H(\tilde{x}_0) = H(x'_0)$

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{h} & X' \\ & \downarrow p & \downarrow p' \\ & \xrightarrow{k} & X \end{array} \quad \begin{array}{l} \exists \text{ lift } h: (\tilde{X}, \tilde{x}_0) \rightarrow (X', x'_0) \quad p' \circ h = p \\ \exists \text{ lift } k: (X', x'_0) \rightarrow (\tilde{X}, \tilde{x}_0) \quad p \circ k = p' \end{array}$$

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{X} \\ & \downarrow p & \downarrow p' \\ & X & \end{array} \quad \begin{array}{l} k \circ h, \text{id}_{\tilde{X}}: (\tilde{X}, \tilde{x}_0) \rightarrow (\tilde{X}, \tilde{x}_0) \\ \text{both lift } \text{id}_X. \end{array}$$

$$p(k \circ h) = (p \circ k) \circ h = p' \circ h = p \circ \text{id}_{\tilde{X}} \quad \text{Hence} \quad k \circ h = \text{id}_{\tilde{X}} \\ \text{and} \quad h \circ k = \text{id}_{X'}$$

Thm $p: \tilde{X} \rightarrow X$, $p': X' \rightarrow X$ are equiv. $\Leftrightarrow H(\tilde{x}_0), H(x'_0)$ are conj. $\in \pi_1(X, x_0)$

If equivalence h exists, w/ $h(\tilde{x}_0) = x'_0 \in X'$

$$H(\tilde{x}_0) = H(x'_0) \text{ conj. to } H(x'_0)$$

If $H(\tilde{x}_0), H(x'_0)$ are conj., $\exists x'_1 \in X'$ s.t. $H(x'_1) = H(\tilde{x}_0)$

Then \exists equiv. h s.t. $\tilde{X} \rightarrow X'$ s.t. $h(\tilde{x}_0) = x'_1$.