

Adjoining a 2-cell

X Hausdorff, $A \subset X$ closed subspace. (both path-connected)

We assume " X " is obtained from A by adjoining a 2-cell.

That is: $D \subset \mathbb{R}^2$: closed unit disk

$\exists f: D \rightarrow X$ cont; $\begin{cases} f|_{\text{int}(D)} \text{ homeo onto } X \setminus A \\ f(\partial D) \subset A. \end{cases}$

Let a be the loop $a = f|_{\partial D} \in \Omega_{a_0}(A)$. $a_0 \in \partial D$ basepoint,

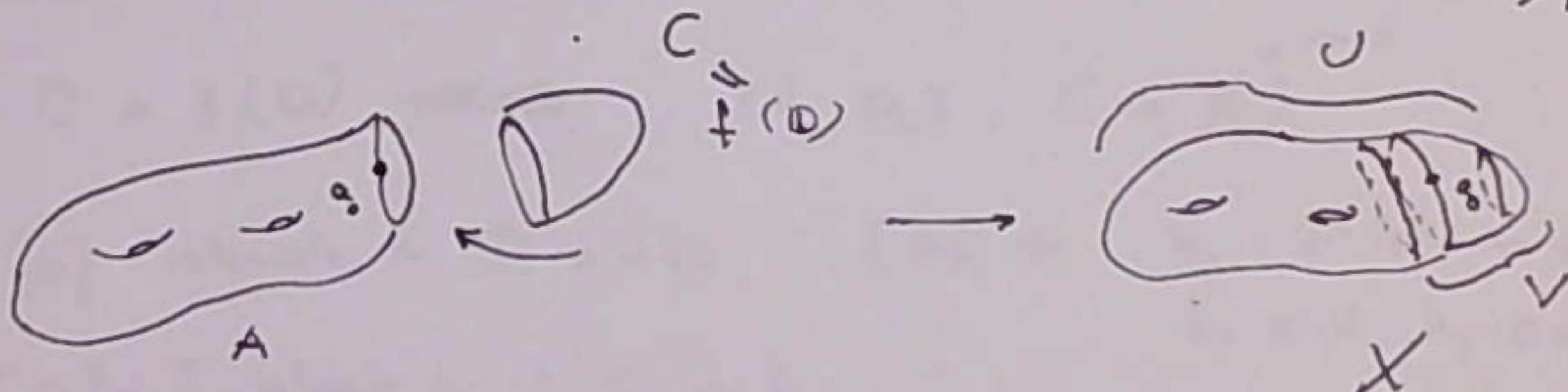
so $[a] \in \pi_1(A, a_0)$

$$a_0 = f(a_0)$$

Prop Let $i_*: \pi_1(A, a_0) \rightarrow \pi_1(X, a_0)$ be induced by inclusion.

Then i_* is onto and $\ker i_* = N \langle [a] \rangle$, the smallest normal subgroup containing $[a]$. Thus $\pi_1(X) \approx \pi_1(A) / N \langle [a] \rangle$.

Ex 1



A is a 2-mfld w/ boundary diff to S^1 , $f|_{\partial D}$ maps ∂D diff. to ∂A .

Consider a tub. nbhd. of $f(\partial D)$ in X , then let $U, V \subset X$ open (as shown). We have $U \cap V$ retracts $\xrightarrow{\text{def.}}$ γ , so $\pi_1(U \cap V) \approx \mathbb{Z}^{(\text{gen'd})}_{\sim [a]}$. $V \sim f(\text{int } D)$, so $\pi_1(V) \approx e_{a_0}$. And U retracts to $\#^k A$, so $\pi_1(U) \approx \pi_1(A)$. By S-vK (since $X = U \cup V$):

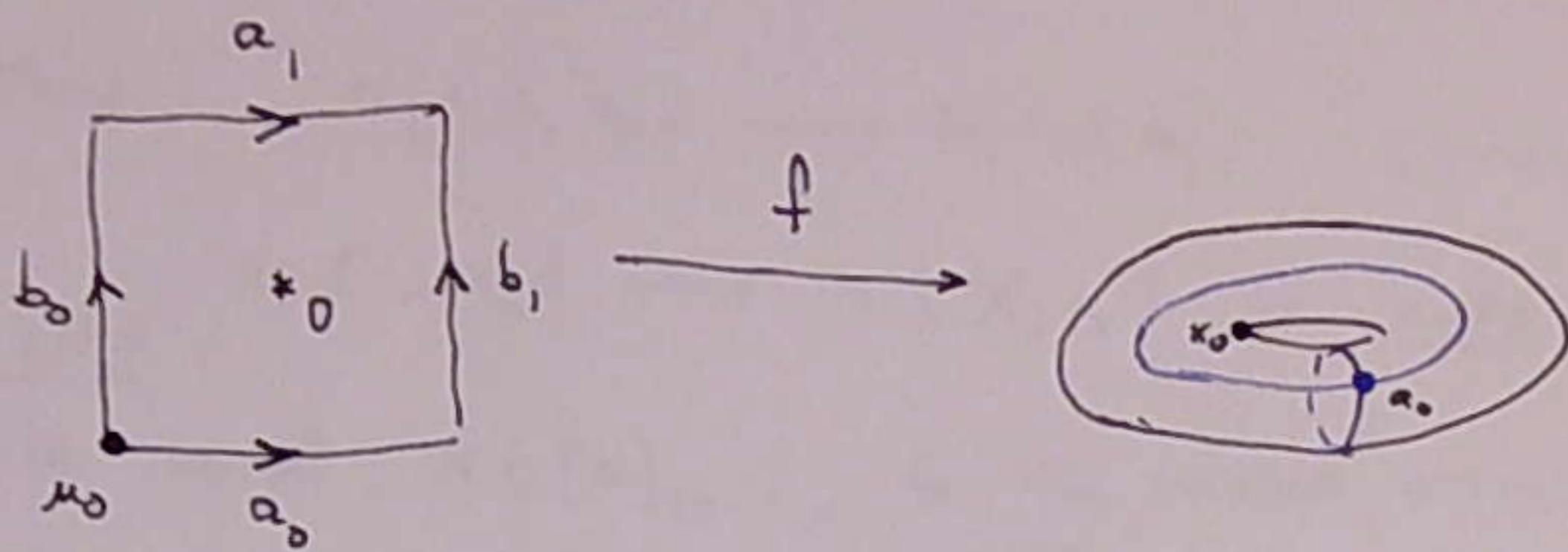
$$\pi_1(A) * \underbrace{\pi_1(V)}_{\approx e_{a_0}} \longrightarrow \pi_1(X) \quad (\text{onto}) \text{ w/ kernel } N \langle [a] \rangle.$$

So

$$\pi_1(X) \approx \pi_1(A) / N \langle [a] \rangle.$$

(2)

Ex. 2



Build T^2 as the quotient space of the square Q ($\approx D^2$ homeo) identifying opposite sides; so the quotient map f maps

$$\begin{aligned} a_0, a_1 &\longrightarrow \alpha \quad (\text{parallel in } T^2) \\ b_0, b_1 &\longrightarrow \beta \quad (\text{meridians in } T^2) \end{aligned} \quad \text{so } f(\partial Q) \sim S^1 \vee S^1$$

$$\text{Here we have } T^2 = X = A \cup f(Q)$$

A is the 'figure 8' $A \sim S^1 \vee S^1$, $f: \text{int } Q \rightarrow V \subset T^2$
 $\cong f(\partial Q)$ (homeo.) open

It is not completely clear how to apply S-vK in this case, hence the proof of the prop is a little more technical.

Step 1

A is a deformation retract of $\cup = X - \{x_0\}$ $x_0 = f(0)$.

Let $C = f(D)$ ~~where~~ (in ex. 2, $C = X$) T^2

$D - \{x_0\}$ retracts to $S^1 = \partial D$ (say via $h_t: D - \{x_0\} \rightarrow D - \{x_0\}$

so $C - \{x_0\}$ retracts to $f(S^1) \subset A$

$$\begin{aligned} h_0 &= \text{id}, h_1: D - \{x_0\} \rightarrow S^1, \\ h_t|_{S^1} &\equiv \text{id}_{S^1} \end{aligned}$$

(via $f \circ h_t$; $f \circ h_1: C - \{x_0\} \rightarrow f(S^1)$)

now

$$\begin{array}{ccc} X - \{x_0\} & = (A - f(S^1)) \sqcup (C - \{x_0\}) & (\text{since } f(S^1) \subset C) \\ \downarrow & \downarrow \text{id} & \downarrow f_t \\ A & A - f(S^1) & f(S^1) \end{array} \quad \text{and } X = A \cup C$$

Thus setting $g_t = \begin{cases} f_t, & C - \{x_0\} \\ \text{id}, & A - f(S^1) \end{cases}$

we retract $X - \{x_0\}$ to $A = (A - f(S^1)) \sqcup f(S^1)$.

③

Thus

$$\pi_1(A, a_0) \longrightarrow \pi_1(U, a_0) \quad (\text{inclusion}) \text{ is } \underline{\text{iso}}.$$

and

we need to show: $\pi_1(U, a_0) \longrightarrow \pi_1(X, a_0)$ is onto,

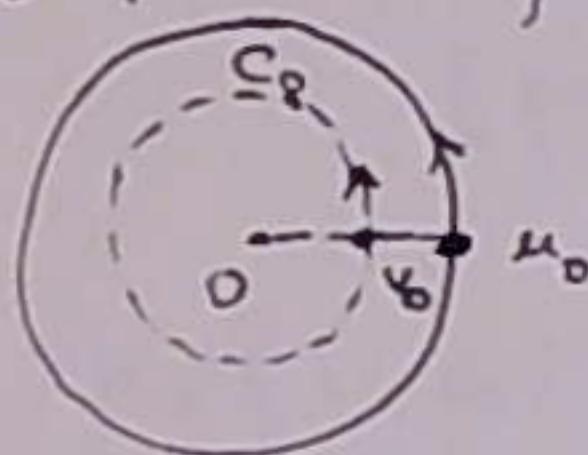
with kernel $N([a]_U)$, to the smallest normal subgroup of $\pi_1(U, a_0)$ containing the homotopy class $[a]$ of the loop $a = f|_{\partial D} \in \Omega_{a_0}(U)$

Step 2 Use the S-vK theorem:

$$X = U \cup V, \text{ where: } U = X \setminus \{x_0\}, V = X \setminus A \\ = f(\text{int } D).$$

$U \cap V = V \setminus \{x_0\}$. Thus $U, V, U \cap V$ are path connected.

(c_0, γ param. \hookrightarrow)



basepoint (note $a_0 \notin U \cap V$) we need a different one!

v_0 : midpoint of radius $[O, w_0] \subset D$.

$f(v_0) = b_0 \in V \setminus \{x_0\}$ is the basepoint.

c_0 : circle through v_0 .

$$\gamma = f \circ c_0 \quad (\text{parametrized})$$

$$\in \Omega_{b_0}(V \setminus \{x_0\})$$

$f : (\text{int } D) \setminus \{O\} \longrightarrow V \setminus \{x_0\}$ is a homeo, so

$\pi_1(U \cap V, b_0)$ is gen'd by $[\gamma]_{U \cap V}$
(htopy class in $U \cap V$).

Since $\pi_1(V, b_0) \approx \{e_{b_0}\}$ (trivial),

S-vK says

$\pi_1(U, b_0) \longrightarrow \pi_1(X, b_0)$ is onto, w/
kernel $N<[\gamma]_U>$ (htopy class in U)

Step 3 Moving the basepoint from b_0 to a_0 .

Let $\delta_0 : [0, 1] \longrightarrow D$ parametrize the radial segment from v_0 to a_0 .

so $f \circ \delta_0 = \delta : [0, 1] \rightarrow X$ is a curve from b_0 to a_0 . Commuting diagram:

$$\pi_1(U, b_0) \longrightarrow \pi_1(X, b_0) \quad (\text{inclusion})$$

$$\text{iso } \hat{\delta} \downarrow$$

$$\downarrow \hat{\delta} \quad \text{iso}$$

$$\pi_1(U, a_0) \longrightarrow \pi_1(X, a_0) \quad (\text{inclusion})$$

Thus

$$\pi_1(U, a_0) \longrightarrow \pi_1(X, a_0) \quad (\text{inclusion}) \\ \text{is } \underline{\text{onto}}$$

(4)

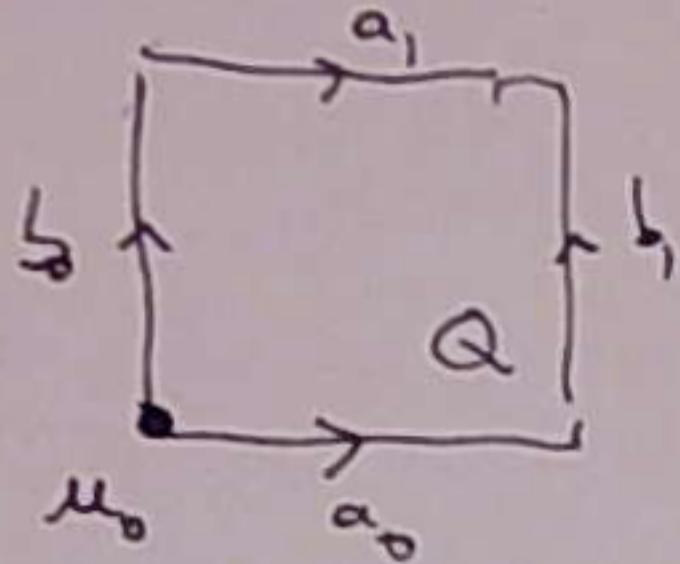
Its kernel is the smallest normal subgroup of $\pi_1(U, a_0)$ containing $\hat{\delta}([\gamma]_U)$, $\gamma = f \circ c_0 \in \Omega_{\delta_0}(U)$

$$\begin{aligned}\hat{\delta}[\gamma]_U &\stackrel{\text{def.}}{=} \left[\underbrace{(\bar{\delta} * \gamma) * \delta_0}_{\in \Omega_{\alpha_0}(U)} \right]_U \\ &= [f \circ (\bar{\delta}_0 * c_0 * \delta_0)]_U = [f \circ \alpha_0]_U,\end{aligned}$$

since $(\bar{\delta}_0 * c_0 * \delta_0) \simeq \alpha_0$ in $\Omega_{\alpha_0}(D)$ (where α_0 parametrizes ∂D on $[0,1]$)
and $(f \circ \alpha_0)([0,1]) \subset A$.
Since U retracts to A , we conclude

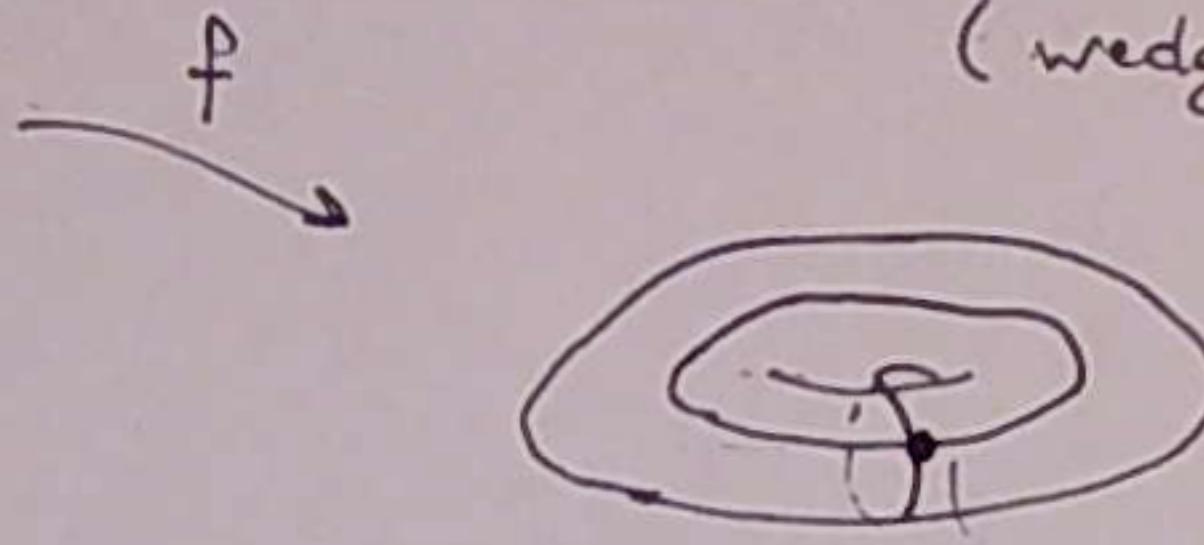
$[f \circ \alpha_0]_U = [\alpha]_U = [\alpha]_A$, so the kernel of
 $\pi_1(U, a_0) \rightarrow \pi_1(X, a_0)$ is the smallest normal subgroup of $\pi_1(U, a_0)$
 containing $[\alpha]_A$. \square

(Ex.)



$$X = \mathbb{T}^2 \quad A = f(\partial Q)$$

(wedge of 2 circles)



$$\pi_1(A, a_0) = F_2$$

$$2Q \xrightarrow{f \circ g}$$

$$\text{generators } [\alpha] = [f \circ a_0], [\beta] = [f \circ b_0]$$

$$\gamma_0 = a_0 \star (b_1 \star (\bar{\alpha}_1 \star \bar{\beta}_0))$$

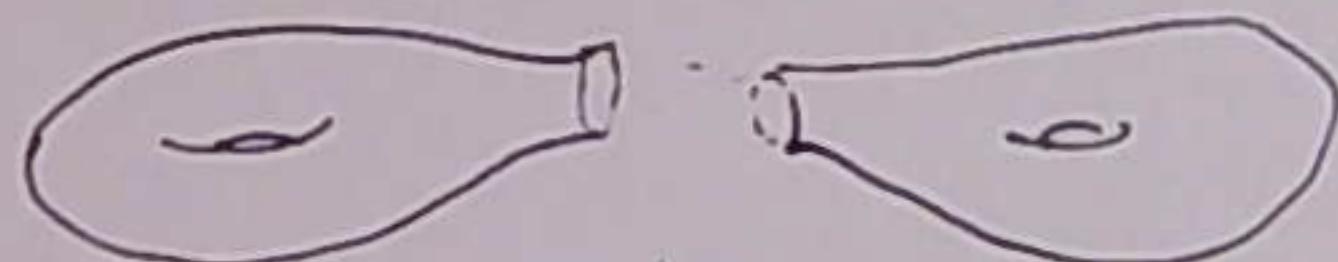
$$[f(\gamma_0)] = \alpha \circ \beta \circ \alpha^{-1} \beta^{-1} = [\alpha, \beta] \quad (\text{commutator in } F_2)$$

$$\pi_1(\mathbb{T}^2) = \frac{F(\alpha, \beta)}{N([\alpha, \beta])} \approx \mathbb{Z}^2 \quad (\text{free abelian group of rank 2})$$

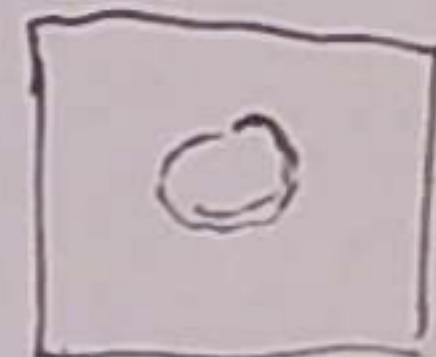
(Ex. 2)

2-hole torus.

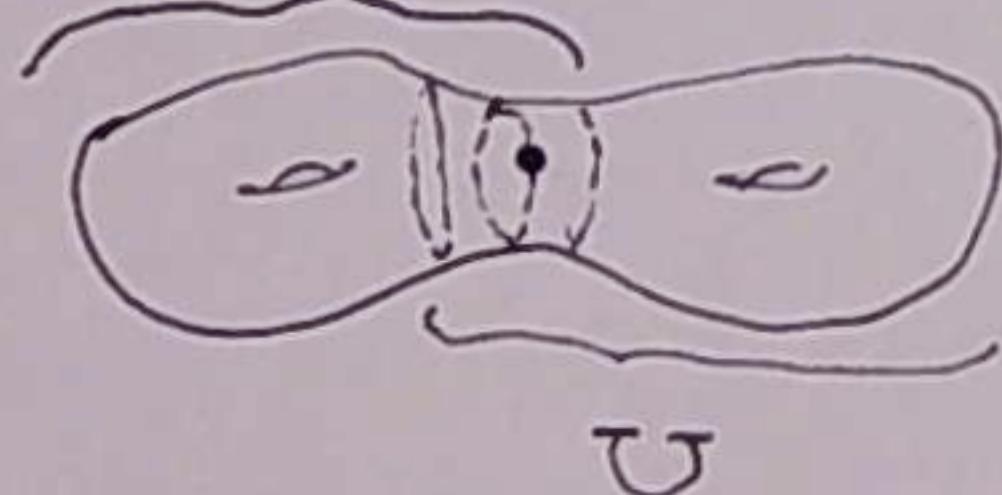
$$\mathbb{T}^2 \# \mathbb{T}^2$$



$$\mathbb{T}^2 \# \mathbb{T}^2$$



$$V \downarrow$$



$$\mathbb{T}^2 \# \mathbb{T}^2 \quad (\text{connection})$$

$$U \cap V$$



eff & g

$$U \text{ def. retracts to } \mathbb{T}^2 \setminus \text{int}(Q) \sim \infty$$

$$\pi_1(U) = F(\alpha, \beta)$$

$$\pi_1(V) = F(\gamma, \delta)$$

new relation corresponds to the core circle in $U \cap V$

$$\alpha \beta \alpha^{-1} \beta^{-1} = \gamma \delta \gamma^{-1} \delta^{-1}$$

$$\pi_1(\mathbb{T}^2 \# \mathbb{T}^2) \approx \frac{F(\alpha, \beta, \gamma, \delta)}{\text{norm} \langle [\alpha, \beta][\gamma, \delta]^{-1} \rangle}$$