

Euler characteristic and vector field singularities

$M$ : compact, connected, oriented mfd w/o boundary

Assume  $M \subset \mathbb{R}^k$ , let  $\epsilon > 0$  small enough so that

$N_\epsilon = \{x \in \mathbb{R}^k \mid \text{dist}(x, M) \leq \epsilon\}$  is a  $k-1$  dimensional mfd

w/ boundary, and  $p: \overset{\circ}{N}_\epsilon \rightarrow M$  (assoc. to  $x \in \text{int } N_\epsilon$  the

unique closest pt in  $M$ ) is a smooth submersion. Let

$g_\epsilon: \partial N_\epsilon \rightarrow S^{k-1}$  (gauss map = unit outward normal  $\hat{n}$ ,  $g_\epsilon(x) = \frac{x - p(x)}{\epsilon}$ )

Then if  $X$  is a vector field on  $M$  w/ finitely many singularities (isolated!)

$z_1, \dots, z_N \in M$ ,  $\sum_i \text{Ind}_M(X; z_i) = \text{deg}(g_\epsilon)$ .

Note the RHS is indep. of the v.f.  $X$ , and the LHS is indep. of  $\epsilon > 0$ . (or  $k$ )

Thus the common value ~~is indep~~ depends only on  $M$ , and we take this as the definition of Euler characteristic,  $\chi_M$ .

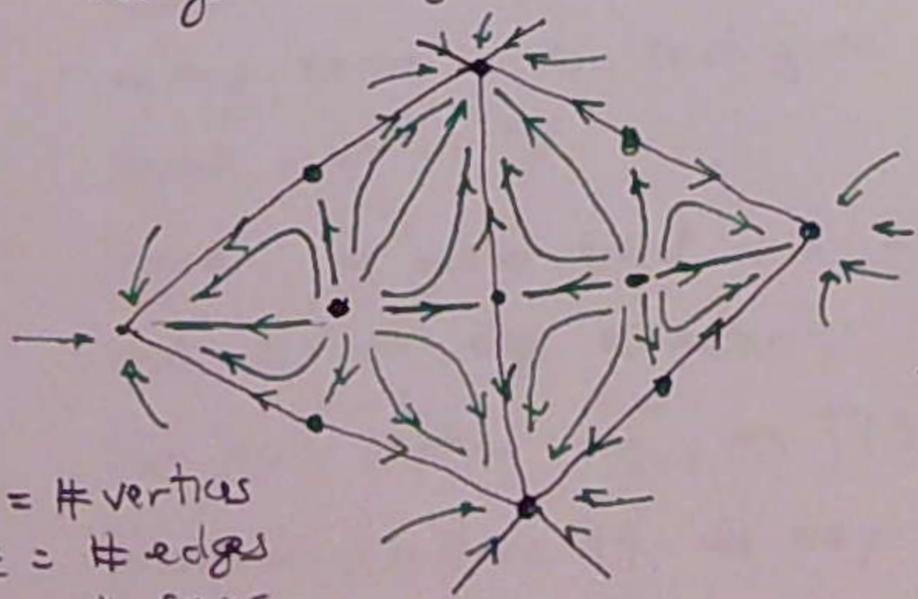
Thus  $\sum_i \text{Ind}_M(X; z_i) = \chi_M = \text{deg}(g_\epsilon)$ , for

any v.f. on  $M$  w/ isolated singularities, and any tubular nbd  $N_\epsilon \subset \mathbb{R}^k$  of  $M$ .

Case of surfaces (informal) Let  $M$  be a cpt. or. connected sfce w/ bdry,

consider a triangulation  $\mathcal{T}$  of  $M$  (two closed triangles intersect only

along an edge, or at a vertex. Associate to  $\mathcal{T}$  a vector field, as follows:



- | source in each face: index +1
- | saddle on each edge: index -1
- | sink on each vertex: index +1

Thus

$$\sum_i \text{Ind}(X_T, z_i) = \chi_M = F - E + V$$

$V = \#$  vertices  
 $E = \#$  edges  
 $F = \#$  faces

Note this shows  $V - E + F \in \mathbb{Z}$  is indep. of the triangulation!  
(since the index sum is indep. of the vector field.)

Exercise If  $M$  is  $S^2$  w/  $g$  handles attached,  $\chi_M = 2 - 2g$ .

Thus If  $M$  admits a v.f. w/o singularities,  $\chi_M = 0$

(in any dimensions) For a surface:  $g=0$ , i.e.  $M = \mathbb{T}^2$  (torus)

The converse is true.

Thm. Let  $M$  be a cpt. connected or. mfd w/o bdy. Suppose  $M$  admits a v.f.  $X$  w/o singularities, then  $\chi_M = 0$ . And if  $\chi_M = 0$ ,  $M$  admits a v.f.  $X$  w/o singularities.

Proof (outline; see G-P p. 146).

Step 1 Let  $V$  be a v.f. on  $\mathbb{R}^k$  w/ finitely many zeros, and  $\sum_i I(V; z_i) = 0$ . Then  $\exists W$  a v.f. on  $\mathbb{R}^k$  without zeros, equal to  $V$  outside a compact set.

Since  $V$  is a map  $V: \mathbb{R}^k \rightarrow \mathbb{R}^k$ , and we consider only non-deg. singularities,  $0$  is a regular value of  $V$ , and  $\exists$  a ball  $B \subset \mathbb{R}^k$ , s.t. the normalized v.f.  $\hat{V}: \partial B \rightarrow S^{k-1}$  has degree 0; and hence extends to the interior of  $B$  as  $W: \mathbb{R}^k \rightarrow \mathbb{R}^k \setminus \{0\}$ , equal to  $V$  outside  $B$ .  
(smooth)

Step 2 On any cpt. manifold  $M \exists$  a vector field w/ only finitely many zeros. In fact given  $U \subset M \exists$  a v.f.  $V$  w/ finitely many zeros, all in  $U$ .

Proof A vector field is a section  $X: M \rightarrow TM$  ( $\pi \circ X = id_M$ ) and a non-deg singularity is a point  $p \in M$  where  $X$  intersects the zero section  $\Sigma_0 \subset TM$  transversally (by a prev. problem). Given any v.f. on  $M$ , we may find a  $C^1$ -close one  $X$  which is transversal to  $\Sigma_0$ .

Thus all singularities of  $X$  (if any) are nondegenerate, hence isolated, hence finite in number (say  $z_1, \dots, z_N$ ).

And then given  $U \subset M$  open we may find a diffeo  $h: M \rightarrow M$ , isotopic to  $id_M$ , so that  $h(\{z_1, \dots, z_N\}) \subset U$  (see [GP, p. 143]; we assume  $\dim M > 1$ ). Then  $X \circ h$  is a v.f. on  $M$  w/ finitely many singularities, all in  $U$ .

Step 3 If  $X_M = 0$ , the v.f.  $V$  from step 2 has all its zeros in  $U$ , w/ indices adding to 0. We may assume  $\varphi: U \rightarrow \mathbb{R}^k$  is ~~an embedding~~ a diffeomorphism, and then  $\tilde{V} = \varphi_* V$  is a v.f. in  $\mathbb{R}^k$  w/ finitely many non-deg. zeros, w/ indices adding to 0.

From step 1,  $\exists$  a v.f.  $\tilde{W}$  in  $\mathbb{R}^k$  w/o zeros, equal to  $\tilde{V}$  outside a precompact open set  $U_1 \subset \mathbb{R}^k$  (meaning  $\bar{U}_1 \subset \mathbb{R}^k$  is compact)

Then defining  $W = \begin{cases} (\varphi^{-1})_* \tilde{W} & \text{in } U, \\ V & \text{in } M \setminus U \end{cases}$ , we see  $W$  is a v.f. on  $M$  w/o zeros, and smooth since  $(\varphi^{-1})_* \tilde{W} = V$  in  $M \setminus \varphi^{-1}(U_1)$

