

Hopf's theorem on maps to S^n

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Thm M^n compact connected oriented, without boundary

$$f, g : M \rightarrow S^n$$

$$\deg(f) = \deg(g) \implies f \simeq g \text{ (homotopic).}$$

Idea of proof The result follows from the Extension Theorem

W^{k+1} compact, connected oriented manifold w/ boundary,

$$f : \partial W \rightarrow S^k \text{ smooth, } \deg(f) = 0 \implies f \text{ extends to } F : W \rightarrow S^k \text{ (smooth).}$$

Indeed the "cylinder" $M \times I$ has boundary $(M \times \{0\}) \sqcup (M \times \{1\})$, and defining a function on $\partial(M \times I)$ to be f at $M \times \{0\}$, g at $M \times \{1\}$, an extension ~~from~~ to $M \times I$ defines a homotopy from f to g (and the degree on the boundary of $M \times I$ is $\deg(g) - \deg(f)$).

In turn the extension theorem follows (Problem 8) from the special case of the theorem : $f : S^k \rightarrow S^k$ smooth, $\deg(f) = 0 \implies f \simeq \text{const.}$ (homotopic).

The following lemmas (of independent interest) are needed :

Lemma 1 $B \subset \mathbb{R}^k$ closed ball, $f : B \rightarrow \mathbb{R}^k$ smooth, $z \in \mathbb{R}^k$ regular value.
 s.t. $f^{-1}(z) \cap \partial B = \emptyset$. Then the number of preimages of z (counted w/
 a sign depending on orientation) equals the "winding number" $W(f|_{\partial B}, z)$
 (the degree of the normalized map to S^{k-1}). [Problem 1/2 in G-P].

Extension lemma 1

$B \subset \mathbb{R}^k$ closed ball, $f : \overline{B^c} \rightarrow Y$ smooth

$f|_{\partial B} \simeq \text{const.} \implies f \text{ extends to } F : \mathbb{R}^k \rightarrow Y.$ [Problem 3 in G-P]

Extension lemma 2

W cpt manifold w/ boundary, $f : \partial W \rightarrow \mathbb{R}^n$ arb. smooth map

Then f admits an extension $F : W \rightarrow \mathbb{R}^n$ (smooth) : $F|_{\partial W} = f$.

[Problem 7 in G-P]

Proof of "special case": $f: S^k \rightarrow S^k$ of degree 0 $\Rightarrow f \simeq \text{const.}$
 (By induction on k : assume true for maps from S^{k-1} to S^{k-1}).

[Prob. B in G-P]

Lemma $f: R^k \rightarrow R^k$ smooth, 0 a regular value.

(Problem 5 in G-P) Ass. $f^{-1}(0)$ is finite, with sign sum $\sum_{x_i \in f^{-1}(0)} \sigma(x_i) = 0$.

Then $\exists g: R^k \rightarrow R^k \setminus \{0\}$ smooth, homotopic to f , coinciding with f outside a compact set.

This follows from the induction hypothesis and the first extension lemma, with $Y = R^k \setminus \{0\}$.

{ Idea to prove the "special case"

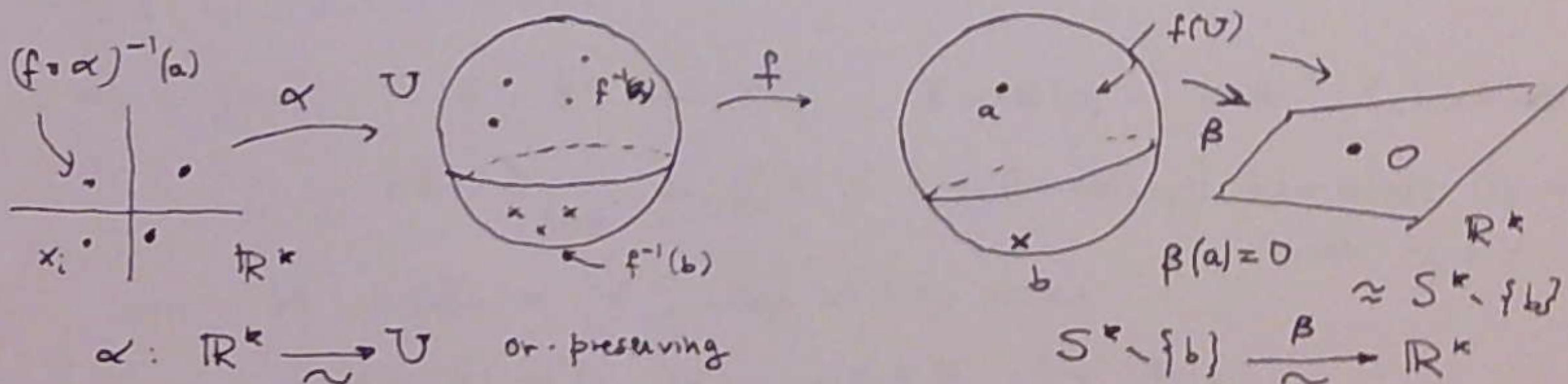
It is enough to find g homotopic to f , and not onto S^k : $g: S^k \rightarrow S^k \setminus \{0\}$.

For that we use the lemma and the idea that $R^k \approx S^k \setminus \{\text{point}\}$.

Proof Let $a \in S^k$ be a reg.-pt. for f ; so $f^{-1}(a)$ is finite (w/ signs adding to 0),

compositions. Fix a second point $b \neq a$ (to play the role of "infinity").

Changing f by composing w/an isotopy (if needed!) find $U \subset S^k$ open, containing $f^{-1}(a)$ and so that $b \notin f(U)$



Let $h = \beta \circ (f|_U) \circ \alpha$ $h: R^k \rightarrow R^k$ smooth on-preserving.

Then 0 is a reg.-value of h , with $\sum_{x_i \in f^{-1}(0)} \sigma(x_i) = 0$.

By the lemma, $\exists \tilde{g}: R^k \rightarrow R^k \setminus \{0\}$ smooth, homotopic to h , equal to h on $R^k \cap C$
 $0 \notin \text{Im}(\tilde{g})$, so $a \notin \text{Im}(\beta^{-1} \circ \tilde{g})$ and $\beta^{-1} \circ \tilde{g}: R^k \rightarrow S^k$ is htopic to $f|_U \circ \alpha$
 $g_1 = \beta^{-1} \circ \tilde{g} \circ \alpha^{-1}: U \rightarrow S^k$, htopic to $f|_U$ but with image omitting $a \in S^k$
 and g_1 coincides w/ $f|_U$ outside $\alpha(C)$, a compact subset of U .

Thus $g = \begin{cases} g_1 & \text{in } U \\ f & \text{in } S^k \setminus \alpha(C) \end{cases}$ is smooth, htopic to f and $g: S^k \rightarrow S^k \setminus \{0\}$, hence is htopic to const.

(3)

From the special case to the general extension theorem (Problem 8)

Given: W^{k+1} cpt w/ bdry, $f: \partial W \rightarrow S^k$ w/ $\deg(f) = 0$
 (connected, oriented)

Use the second extension lemma to extend f to $F: W \rightarrow \mathbb{R}^{k+1}$

Since 0 is trivially a regular value for $F|_{\partial W}$, by transversality extension
 we may assume 0 is a reg. value for F , with finite preimage.

Comparing F w/ an isotopy, we may also assume $F^{-1}(0)$ is contained
 supported away from ∂W

in a ball B inside W , with $W(F|_{\partial B}; 0) = 0$ (by homotopy invariance)

and the fact $F: W \setminus B \rightarrow \mathbb{R}^{k+1} \setminus \{0\}$ sat. $F|_{\partial W} = f$, of winding number 0.

But this means $\deg \hat{f}|_{\partial B} = 0$, where $\hat{f}: \partial B \rightarrow S^k$ is the normalization.

Thus $F|_{\partial B}$ is homotopic to a constant (by the special case); and

now the ^{first} extension lemma says $F|_{\partial B}$ extends smoothly to $\hat{F}: \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1} \setminus \{0\}$,

and in particular to $\tilde{F}: W \rightarrow \mathbb{R}^{k+1} \setminus S^k$ (after normalization), where the
 values of $\tilde{F}|_{\partial W} = F|_{\partial W}$ have not been changed.

Proof of extension lemma

We're given $f: \bar{B}^k \rightarrow Y$, $B = B(0, R) \subset \mathbb{R}^k$ (closed ball)

s.t. $f|_{\partial B} \simeq y_0 \in Y$: (smoothly) homotopic to a
 const, $y_0 \in Y$.

via $H: \partial B \times I \rightarrow Y$, where we may assume

$$H(x, t) = \begin{cases} f|_{\partial B}(x), & x \in \partial B, t \in [0, 1] \\ y_0, & x \in \partial B, t \in [0, 1/2]. \end{cases}$$

Then define the extension $F: B_R(0) \rightarrow Y$ by:

$$F(x) = \begin{cases} H\left(\frac{Rx}{\|x\|}, \frac{\|x\|}{R}\right), & x \neq 0, \|x\| \leq R \\ y_0, & x = 0 \end{cases}$$

note $F(x) = y_0$, $\|x\| < R/2$ and $F(x) = f\left(\frac{Rx}{\|x\|}\right)$, $\frac{3R}{4} < \|x\| \leq R$,

thus the extension of F as f (outside of B) defines $F: \mathbb{R}^k \rightarrow Y$
 smooth.