

Index of isolated vector field singularities

$U \subset \mathbb{R}^m$ $v: U \rightarrow \mathbb{R}^m$ smooth vector field w/ isolated singularity at $z \in U$. $v(z) = 0$

$$\hat{v}: \partial B \xrightarrow{\sim} S^{m-1}$$

$$\hat{v}(x) = \frac{v(x)}{\|v(x)\|}, \quad B \subset \bar{U} \quad (B: \text{small ball w/ center } z)$$

$$\text{Ind}(v, z) = \deg(\hat{v}|_{\partial B}) \quad [\text{check: map of choice of } B]$$

Thm

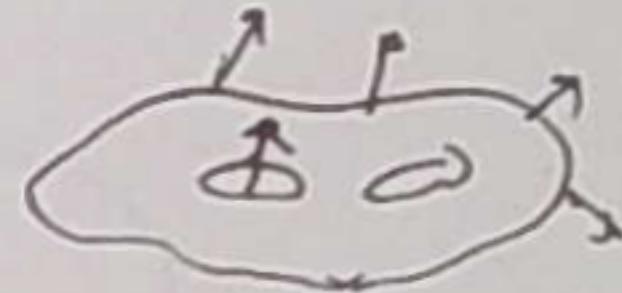
(Hopf).

$X^m \subset \mathbb{R}^m$ compact manifold w/ body

Gauss map:

$$g: \partial X \rightarrow S^{m-1}$$

(unit outward normal)

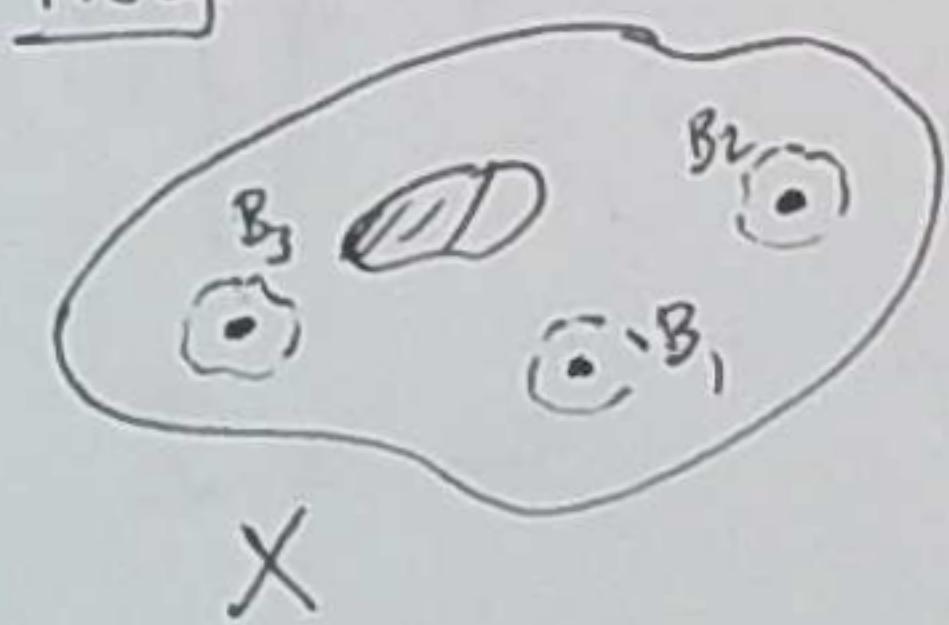


$v: v.f.$ in X w/ isolated singularities, v outward on ∂X

Then $\sum_{v(z_i)=0} \text{Ind}(v, z_i) = \deg g$ In part. independent of v

(ex: $X = \mathbb{D}^m$ (disk) $\Rightarrow \sum_i \text{Ind}(v, z_i) = 1$)

Proof



Remove a small ball B_i at each z_i , to get a mfld w/ body $X \setminus (\cup B_i) = \hat{X}$.

$$v|_{\partial \hat{X}}: \partial \hat{X} \rightarrow S^{m-1}$$

$$\deg(\hat{v}|_{\partial \hat{X}}) = 0$$

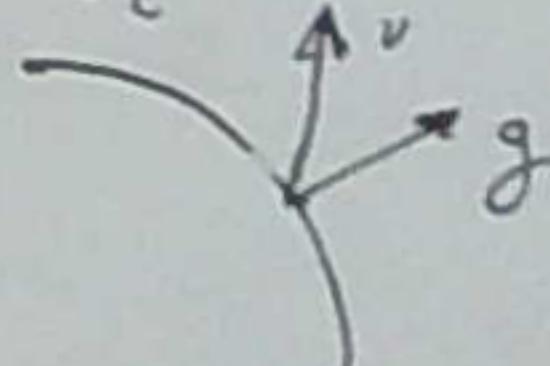
since \hat{v} extends to \hat{X}

$$\partial \hat{X} = \partial X - (\partial B_1 \sqcup \partial B_2 \sqcup \dots \sqcup \partial B_n)$$

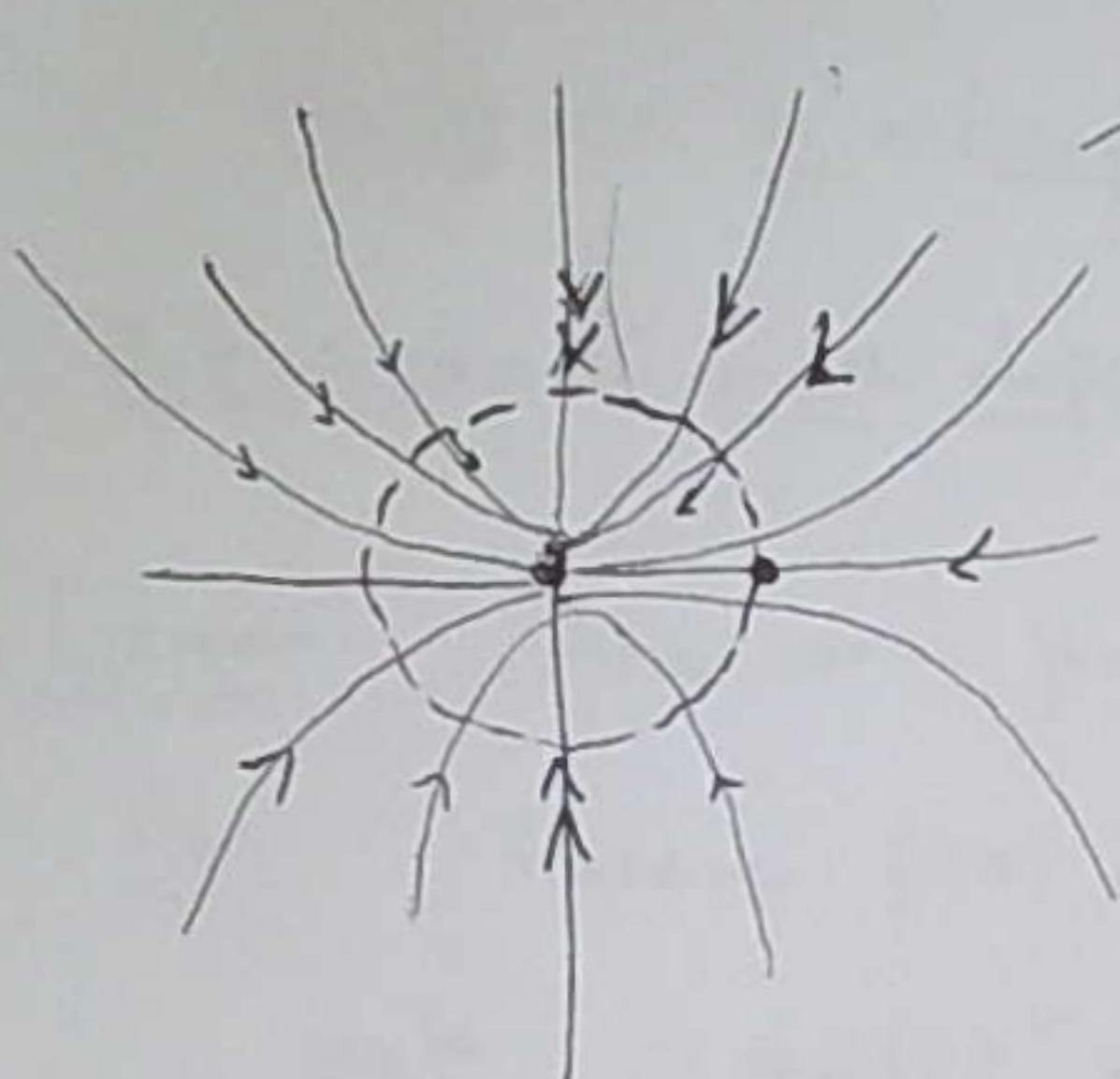
$\Rightarrow \text{Ind}(\hat{v}|_{\partial \hat{X}}) = \deg(\hat{v}|_{\partial \hat{X}}) - \sum_{z_i} \text{Ind}(v, z_i)$

$$\hat{v}|_{\partial \hat{X}} \simeq g|_{\partial \hat{X}}$$

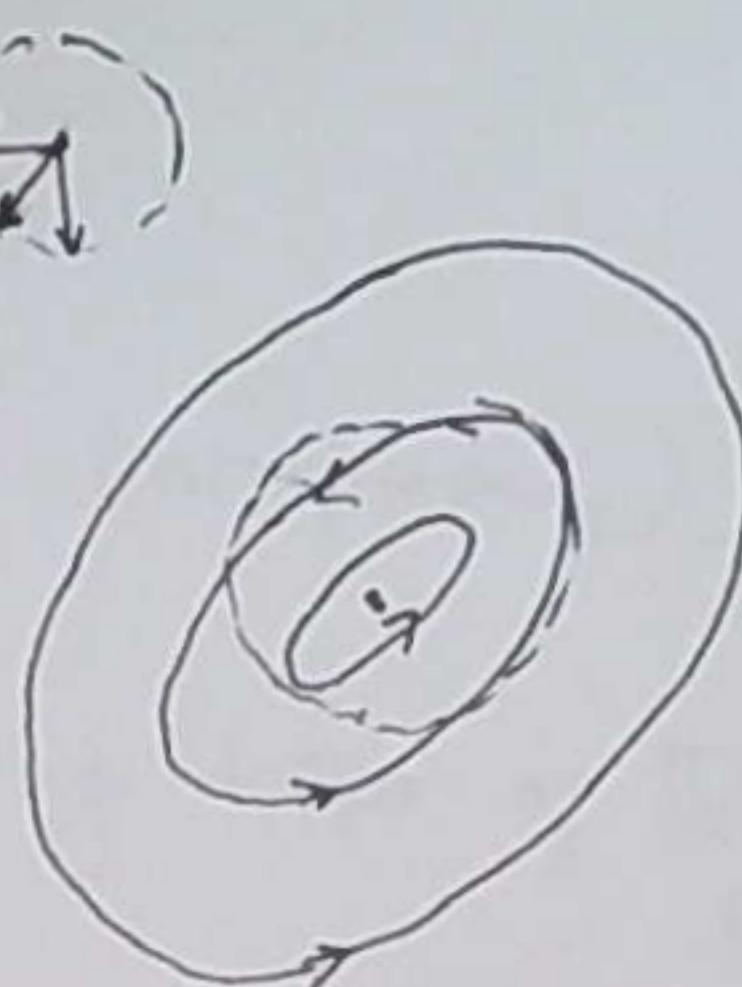
$$\text{so } \deg(\hat{v}|_{\partial \hat{X}}) = \deg(g|_{\partial \hat{X}}).$$



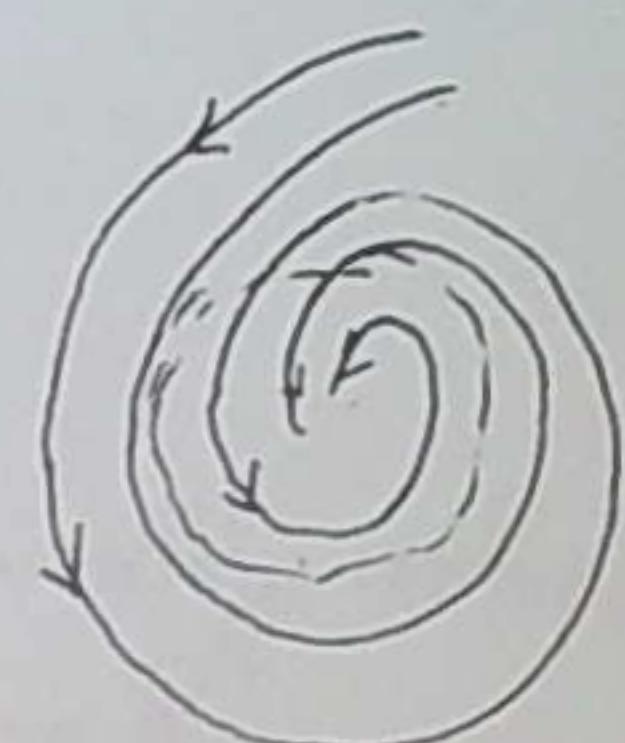
$(1-t)v + tg$
is outward $\forall t \in [0, 1]$
since $\langle v, g \rangle > 0$



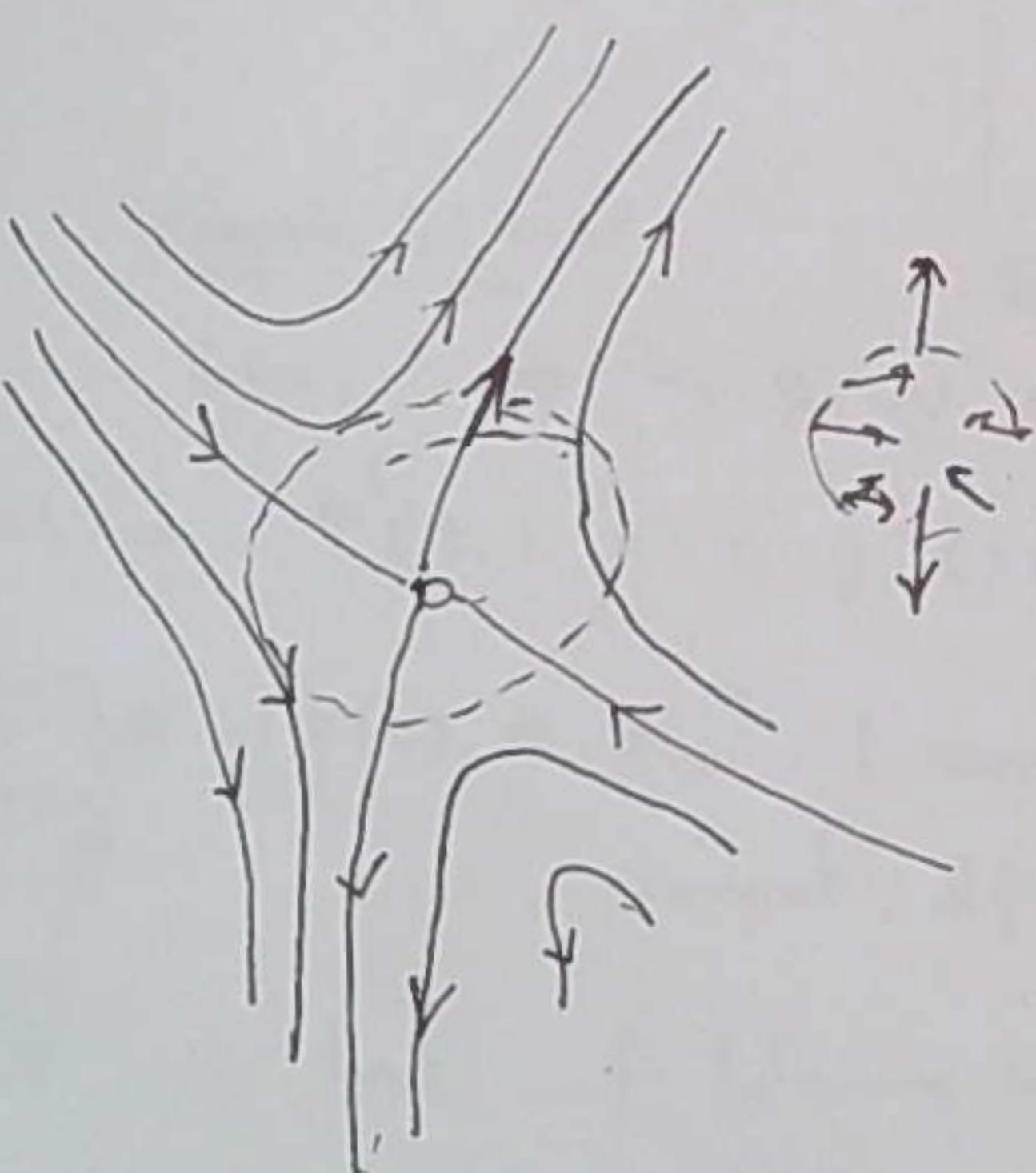
$$I = -1$$



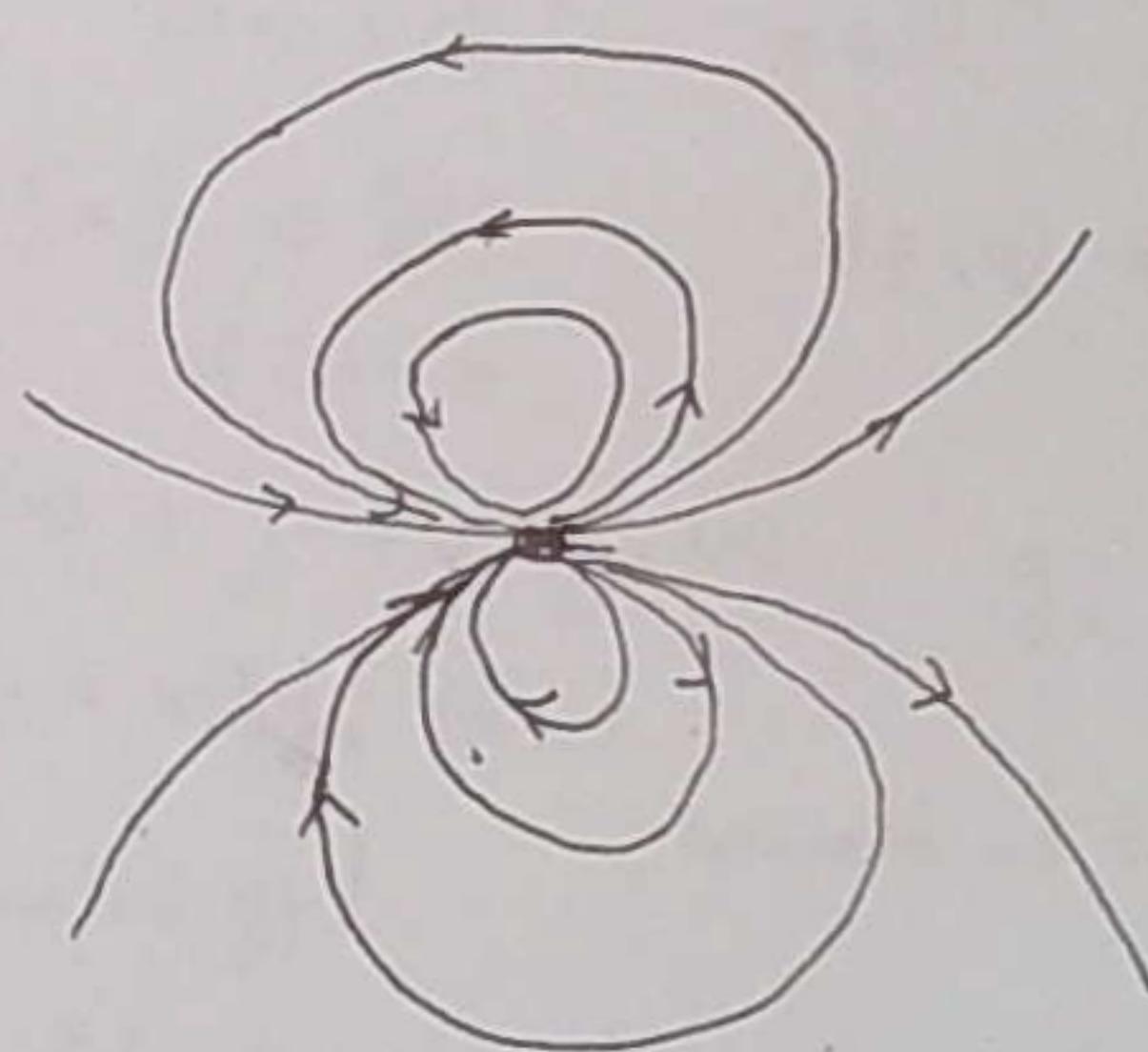
$$I = 1$$



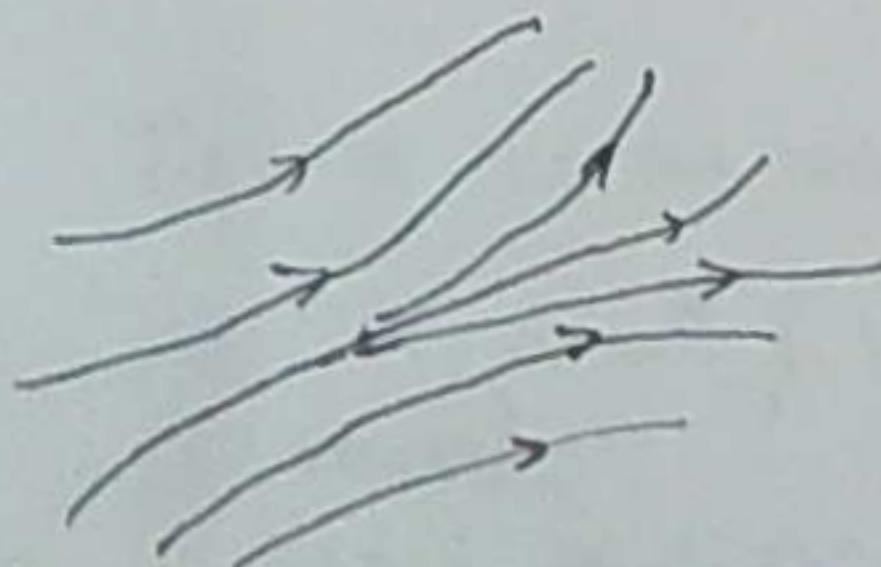
$$I = 1$$



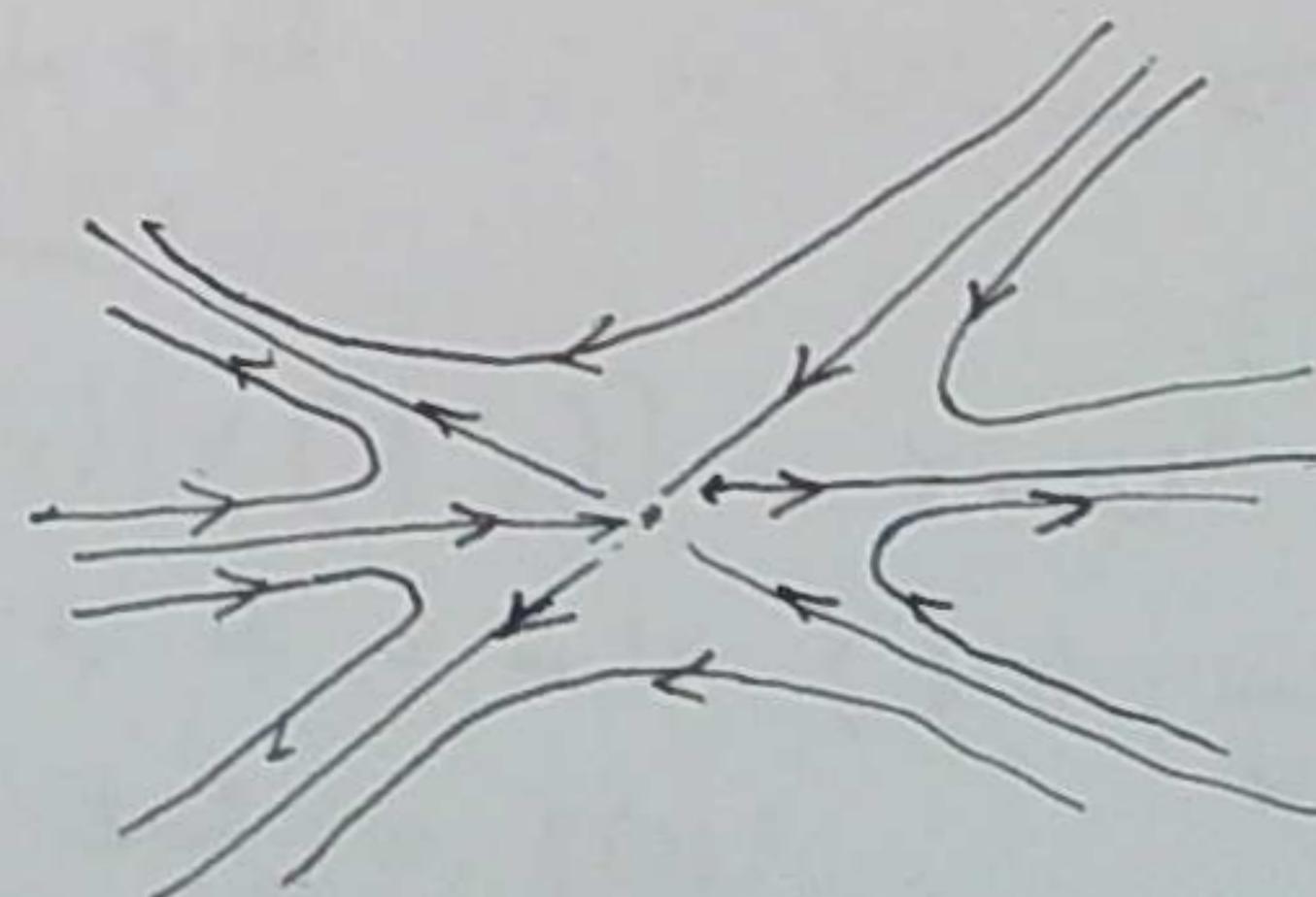
$$I = -1$$



$$I = 2$$



$$I = 0$$



$$I = -2$$

$\underline{z^k}$: index k

\bar{z}^k : index $-k$

$k \in \mathbb{Z}$

deg.
saddle

Extension to manifoldsInvariance under diffeos. (or-preserv.).

① Lemma

Any or.-preserving diffeo. of \mathbb{R}^n is smoothly isotopic to idPf. Assume $f(0) = 0$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

note $df(0)[x] = \lim_{t \rightarrow 0} \frac{f(tx)}{t} \quad x \in \mathbb{R}^n$

Define an isotopy $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$F(t, x) = \begin{cases} f \frac{(tx)}{t} & 0 < t \leq 1 \\ df(0)[x] & t = 0 \end{cases} \quad x \in \mathbb{R}^n$$

smooth at $t=0$?

$$f: \mathbb{R}^n \rightarrow \mathbb{R} \quad f(0) = 0$$

$$f(x) = x_1 g_1(x) + x_2 g_2(x) + \dots + x_n g_n(x)$$

$$i=1, \dots, n \quad F^i(t, 1) = x_1 g_i^i(tx) + \dots + x_n g_n^i(tx) \quad g_i: \mathbb{R}^n \rightarrow \mathbb{R}$$

smooth

For proof $\Rightarrow df(0) \in GL_n^+$ (connected matrix grp.) (smooth at $t=0$).connect $df(0)$ to Id by invertible, pos. lin. transf.

②

Let $f: U \rightarrow U'$ diff ($U, U' \subset \mathbb{R}^n$) $v: U \rightarrow \mathbb{R}^n$ vector field , $w: U' \rightarrow \mathbb{R}^n$ v.f.suppose v, w are "f-related".

$$w(y) = df(f^{-1}y)[v(f^{-1}y)] \quad y \in U'$$

If z is an isolated sing'ty of v in U ($f(z)$: isolated sing'ty of w in U')

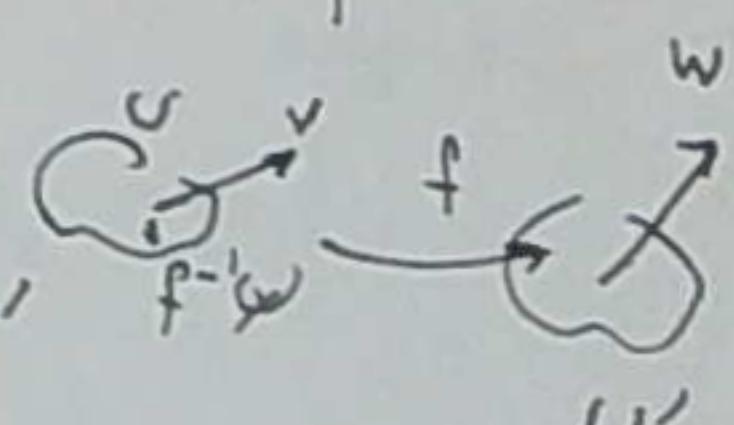
$$\Rightarrow \text{End}_U(v; z) = \text{End}_{U'}(w, f(z)).$$

(or convex)

Proof Ass. $z = 0 = f(z)$

$$(f_t)_{t \in [0,1]}: U \rightarrow U_t \subset \mathbb{R}^n$$

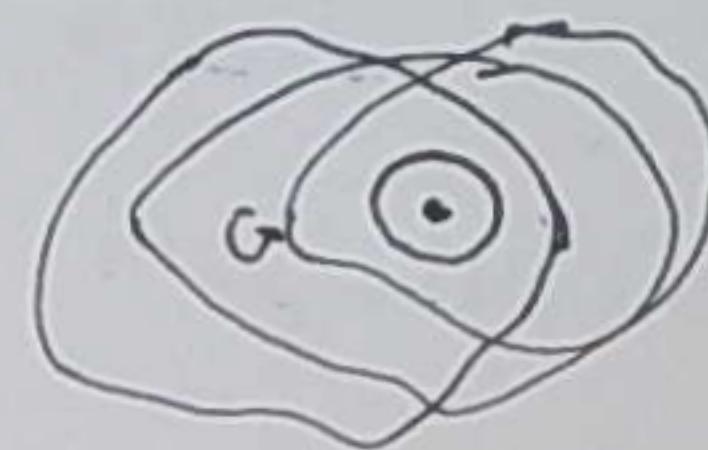
smooth family of embeddings (diffeo onto image)



Let $v_t(y) = df_t|_{f_t^{-1}(y)} [v(f_t^{-1}y)]$: v.f. on $U_t = f_t(U)$

f_t - related to v

v_t : defined and nonzero (except at 0)
in a small ball w/ center 0.



$$\begin{cases} f_0 = \text{id} \\ f_1 = f \end{cases}$$

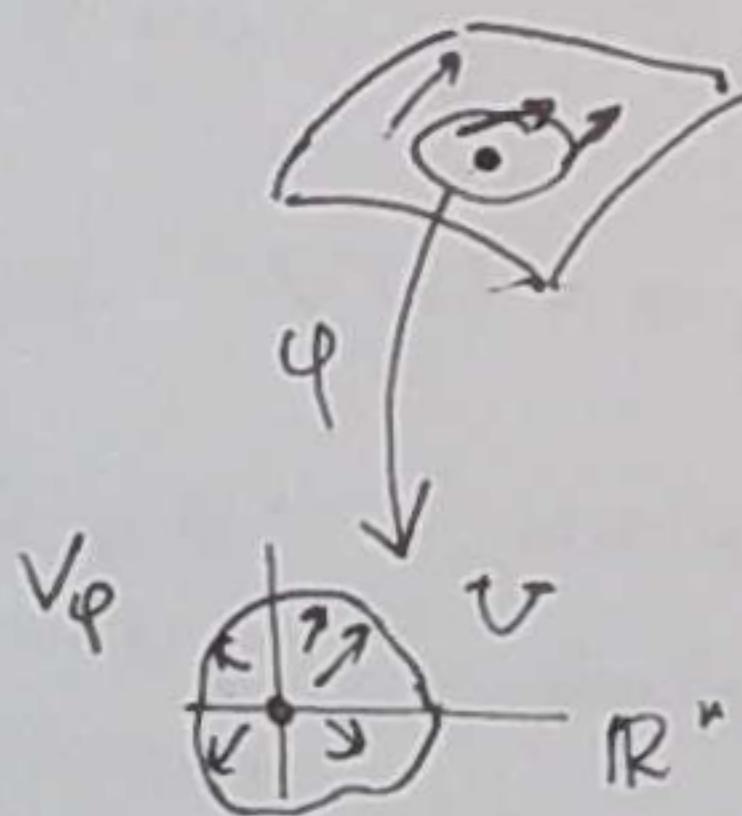
$\hat{v}_t|_{\partial B}$ are all homotopic. $\Rightarrow \text{Ind}(\hat{v}_0; 0) = \text{Ind}(v_1, 0)$

□

[Rk.] If v is not positive consider $\rho \circ f \circ \rho^{-1}$ (ρ : reflection on some hyperplane through 0)

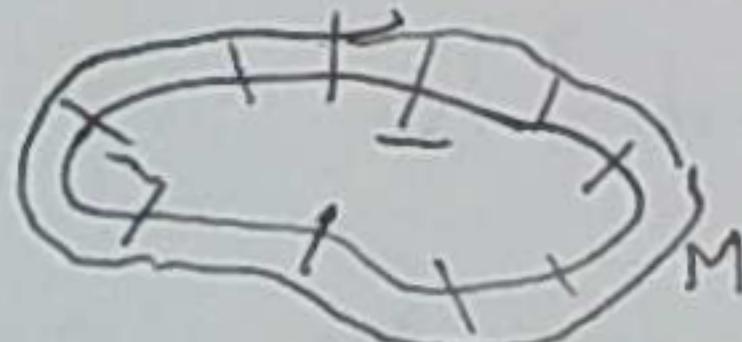
Thus one extends defines $\text{Ind}(v; z)$

(v : v.f. on M , manifold, z : isolated sing.) via local charts.



$\text{Ind}(v; z) = \text{Ind}(v_\phi; 0)$
(indep. of chart ϕ , /
by diffeo-invariance) $U \subset \mathbb{R}^n$

Extension of Hopf's theorem M : closed manifold $\partial M = \emptyset$



Assume $M \subset \mathbb{R}^k$ V : v.f. on M w/ isolated singularities.

$$N_\varepsilon = \{x \in \mathbb{R}^k \mid d(x, M) < \varepsilon\}$$

$\Gamma: N_\varepsilon \rightarrow M$ $\Gamma(x) = \text{closest pt on } M \text{ to } x$

Tub. Nbd thm For $\varepsilon > 0$ small enough. $\Gamma: N_\varepsilon \rightarrow M$ is smooth,
and is a submersion.

Gauss map on ∂N_ε : outward unit normal $g: \partial N_\varepsilon \rightarrow S^{n-1}$

[Thm] $\sum_i \text{Ind}(v; z_i) = \deg(g|_{\partial N_\varepsilon})$ $\xrightarrow{\text{Indep. of } v}$
where: \sum_i (v w/ non-deg. zeros) $\text{Ind}(v; z_i)$
extend v to N_ε , use Hopf's thm in \mathbb{R}^n .

(5)

Proof of theorem (idea: extend ∇ to a v.f. W in N_ε and use the theorem for bdd submanifolds of \mathbb{R}^k)

$x \in N_\varepsilon$, $r(x) \in M$: closest point, $r: N_\varepsilon \rightarrow M$ smooth submersion
 $x - r(x) \in (T_x M)^\perp \in \mathbb{R}^k$ (ε small).

Consider $\varphi(x) = \|x - r(x)\|^2$, $\varphi: N_\varepsilon \rightarrow [0, \varepsilon^2]$
 $\text{grad } \varphi(x) = 2(x - r(x))$

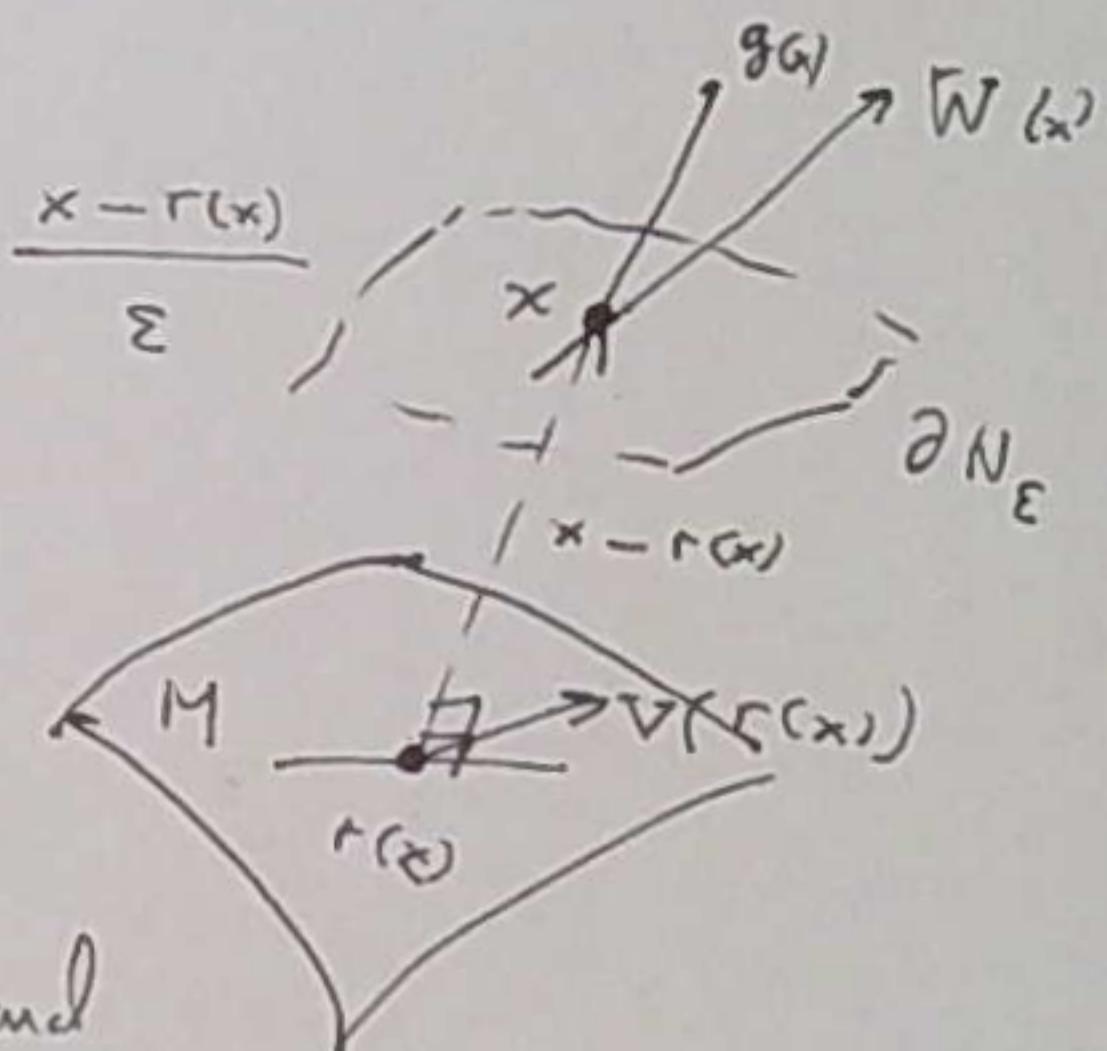
Hence if $x \in \partial N_\varepsilon$: $g(x) = \frac{\text{grad } \varphi(x)}{\|\text{grad } \varphi(x)\|} = \frac{x - r(x)}{\varepsilon}$

Extend ∇ to a v.f. W on N_ε , via:

$$w(x) = \underbrace{(x - r(x))}_{(T_{r(x)} M)^\perp} + \underbrace{\nabla(r(x))}_{T_{r(x)} M}$$

$$\langle w(x), g(x) \rangle = \varepsilon > 0 : W \text{ points outward}$$

Ass. $\nabla(z) = 0$, $z \in M$



$$\begin{aligned} h \in \mathbb{R}^k & \quad dw(x)[h] = d\nabla(r(x))dr(x)[h] + h - dr(x)[h] \\ x \in N_\varepsilon & \\ z \in M & \quad dw(z)[h] = d\nabla(z)dr(z)[h] + h - dr(z)[h] \\ & = \begin{cases} d\nabla(z)[h], & h \in T_z M \quad (\text{since } dr(z)[h] = h) \\ h, & h \in (T_z M)^\perp \quad (\text{since } dr(z)[h] = 0) \end{cases} \end{aligned}$$

Thus $\det \left[dw(z) \right]_{\mathbb{R}^k} = \det \left[d\nabla(z) \right]_{T_z M}$

So $i(W, N_\varepsilon, z) = i(\nabla, M, z)$ (since we assume the zeros are non-deg.) and the thm follows from Hopf's theorem in \mathbb{R}^k

Extensions (see [Milnor])

- vector fields w/ degenerate zeros.
- mflds w/ bdy (v pointing outward at ∂M).

[note for non-deg zeros
 $I(\nabla, z) = \pm 1$]

according to sign $\det [d\nabla(z)]$