

Brouwer degree / applications

M, N oriented manifolds (w/o bdy), $\dim M = \dim N = n$
 (M compact) N connected.

$f: M \rightarrow N$ diff'ble

If $x \in M$ is a reg. pt. $\Rightarrow df(x) \in L(T_x M, T_{f(x)} N)$ ISO
 $\text{sign } df(x) = \pm 1$

$y \in N$ reg. value

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \text{sign}(df(x)) \in \mathbb{Z}$$

loc. const. over reg. values y ; def. for y in an open dense set of N .

Thm. A $\deg(f; y)$ does not depend on the reg. value y
 ($\Rightarrow \deg(f) \in \mathbb{Z}$ defined).

Thm B f smoothly homotopic to $g \Rightarrow \deg(f) = \deg(g)$.

(For $f: M^n \rightarrow S^n$: saying $\deg f = \deg g \Rightarrow f$ htopic to g) Hopf's thm

Lemma 1 X^n cpt oriented mfd w/ bdy, $\partial X = M^{n-1}$

Suppose $f: M \rightarrow N$ extends to a smooth $F: X \rightarrow N$.

(N conn;
or. $\dim N = n-1$)

Then $\deg(f; y) = 0 \quad \forall y \in N$ reg. value.

(Re: the converse is true if $N = S^{n-1}$)

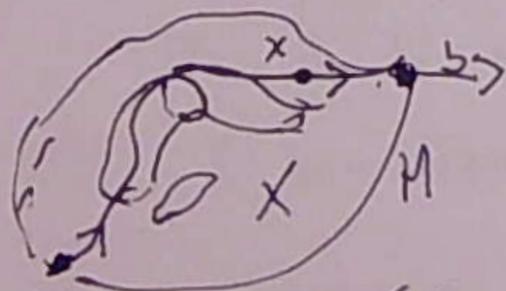
Pf of Lemma 1

(i) Assume $y \in N$ is a reg. value for F and f . Then $F^{-1}(y)$ is

a 1-dim'l submanifold of X : a union of circles and arcs w/ bdy in $\partial X = M$

Let $A \subset F^{-1}(y)$ be an arc, $\partial A = \{a, b\} \subset M$

Claim $\text{sign } df(a) + \text{sign } df(b) = 0$ (prove lemma in this case)



orient $T_x A$ by requiring $(v_1, \dots, v_n) dF(x)$
 to take (v_2, \dots, v_n) to a pos. basis of $T_y N$.

(v_1, \dots, v_n) : pos. basis at $T_x X$, $v_1 \in T_x A$ positive.

$x \in A \quad dF(x)[v_1] = 0$

At the endpoints

$v_1(a)$: inward, $v_1(b)$: outward

If $(v_1(b), v_2(b), \dots, v_n(b))$ is positive.

then $(v_1(a), v_2(a), \dots, v_n(a))$ is negative.

[$dF|_a$ maps $v_2(a), \dots, v_n(a)$ to a pos. basis of $T_a N$]

so $df(a) = dF(a)|_{\langle v_2(a), \dots, v_n(a) \rangle} \Big|_{T_a M}$ and $df(b) = dF(b)|_{T_b M}$ have opposite signs.
(proves claim).

(ii) If $\gamma_0 \in N$ reg. value for f , but not for F .

$\deg(f; \gamma)$ const for γ a reg. value in a nbd $U \subset N$ of γ_0 .

Choose a reg. value $\gamma \in U$ for F .

Then $\deg(f; \gamma_0) = \deg(f; \gamma) = 0$ since γ is also reg. value for F .

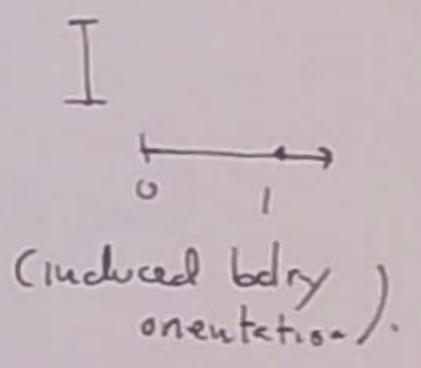
Lemma 2 Let $F: [0, 1] \times M \rightarrow N$ smooth homotopy.

$F(0, x) = f(x), F(1, x) = g(x) \quad f, g: M \rightarrow N.$

Claim $\deg(g; \gamma) = \deg(f; \gamma)$ for any common reg. value $\gamma \in N$.

Pf. $[0, 1] \times M$: oriented as a product

$\partial([0, 1] \times M) = (\{0\} \times M) \sqcup (\{1\} \times M)$
 $F = f \rightarrow \underbrace{\quad}_{-\mathcal{O}_M} \quad \underbrace{\quad}_{\mathcal{O}_M}$



From Lemma 1 $\deg(F|_{\partial([0, 1] \times M)}; \gamma) = 0$

$\deg(g; \gamma) - \deg(f; \gamma) = 0$

Lemma 3 (Isotopy thm.)

N : smooth oriented mfd. $\gamma, z \in \text{int}(N)$

Then $\exists h: N \rightarrow N$ (diffeo) smoothly isotopic to id_N , s.t. $h(\gamma) = z$.

isotopic (homotopic, the $F_t: N \rightarrow N$ $t \in [0, 1]$ are all diffeomorphisms)

Cor Generalizes to $\{\gamma_1, \dots, \gamma_n\} \in N, \{z_1, \dots, z_n\} \quad h(\gamma_i) = z_i$ (if $\dim N > 1$)

Pf of thm A

$f: M \rightarrow N$ smooth y, z reg. values
 $h: N \rightarrow N$ diffeo. $h \approx \text{id}_N$ $h(y) = z$ (h preserves or.)
 $\text{deg}(f; y) = \text{deg}(h \circ f, h(y)) = \text{deg}(h \circ f, z) = \text{deg}(f, z)$
 \downarrow $(h \circ f)^{-1}(h(y)) = f^{-1}(y)$ \uparrow Lemma 2
 h or. preserving.

Pf of thm B

Ass f homotopic to g , let y be a common reg. value.
 $\text{deg } f = \text{deg}(f; y) \stackrel{\text{Lemma 2}}{=} \text{deg}(g; y) = \text{deg}(g)$.

Appl'n 1

$f: M \rightarrow N$ diffeo. (M, N oriented)
 M cpt, w/o bdy.
 f positive $\rightarrow \text{deg } f = +1$
 f negative $\rightarrow \text{deg } f = -1$

f orient. reversing $\rightarrow f$ is not homotopic to id_N .
 (ex: antipodal map $\alpha: S^n \rightarrow S^n$, n even)

Appl'n 2

S^n admits a non-vanishing (tangent) vector field
 $\iff n$ is odd.

$n = 2k-1$ $S^{2k-1} \hookrightarrow \mathbb{R}^{2k}$
 $v(x) = (x_2, -x_1, x_4, -x_3, \dots, x_{2k}, -x_{2k-1})$
 $\langle v(x), x \rangle_{\mathbb{R}^{2k}} = 0$

Suppose $v(x)$ is a nonvanishing v.f. on S^n .

Normalizing v , we get a map $v: S^n \rightarrow S^n$.

Then let $F: S^n \times [0, \pi] \rightarrow S^n$ be given by

$F(x, \theta) = (\cos \theta)x + (\sin \theta)v(x)$.

$F(x, 0) = x$

$F(x, \pi) = \alpha(x)$

$\|F\|^2 = \cos^2 \theta \|x\|^2 + 2 \sin \theta \cos \theta \underbrace{\langle x, v(x) \rangle}_0 + \sin^2 \theta \underbrace{\|v(x)\|^2}_1 = 1$

$\text{id}_{S^n} \cong \alpha \implies n$ is odd (see Appl'n 1).