

# Orientable manifolds

/  $\dim E = n \geq 1$

( $\dim E = 0$ :  
 $\pm 1$ )

①  
4/19

① Orientation  $\mathcal{O}$  on a fin.dim. v.sp.  $E$  : equiv. class ordered bases

$E = (v_1, \dots, v_n)$  ordered basis

$E \equiv F$  if the transition matrix has positive det.

-  $\mathcal{O}$  is the other equiv. class.

Recall  $GL_n^+$  is path-connected : any two bases in  $\mathcal{O}$  can be joined by a path of bases in  $\mathcal{O}$ .

$T \in L(E, F)$  iso par. if it takes bases in  $\mathcal{O}_E$  to bases in  $\mathcal{O}_F$ .

② Orientable manifold:

$M$ : diff'ble mfld (possibly w/ bdry).

orientation an assignment  $x \mapsto \mathcal{O}_x$  : orientation of  $T_x M$ .

s.t. any  $\varphi: U \rightarrow (\mathbb{R}^n, \text{std})$  chart is positive

( $\exists$  an atlas w/ positive transition maps.

Ex. 1 Co-induced orientation  $f: M \rightarrow N$  local diff  $\mathcal{O}'$  orientation

$M$  can be oriented by requiring  $df(x) \in \text{Iso}(T_x M, T_{f(x)} N)$  to be positive.

Ex. 2 Induced orientation

$f: M \rightarrow N$  local diff, onto and  $M$  connected

Prop.  $N$  is orientable  $\Leftrightarrow \forall x_1, x_2 \in M \quad f(x_1) = f(x_2)$  and oriented

$df^{-1}(f(x_2)) df(x_1) : T_{x_1} M \rightarrow T_{x_2} M$  is positive

Prop.  $M, N$  oriented w/ same dimension.  $f: M \rightarrow N$  local diff

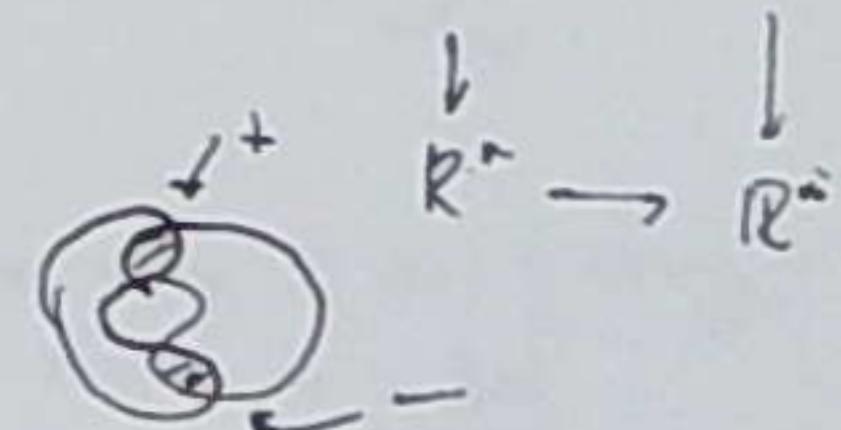
$\Rightarrow \{x \in M \mid df(x) \text{ is positive}\}$  is open in  $M$ .  $U \xrightarrow{f} V$

- On a connected mfld only two orientations are possible.

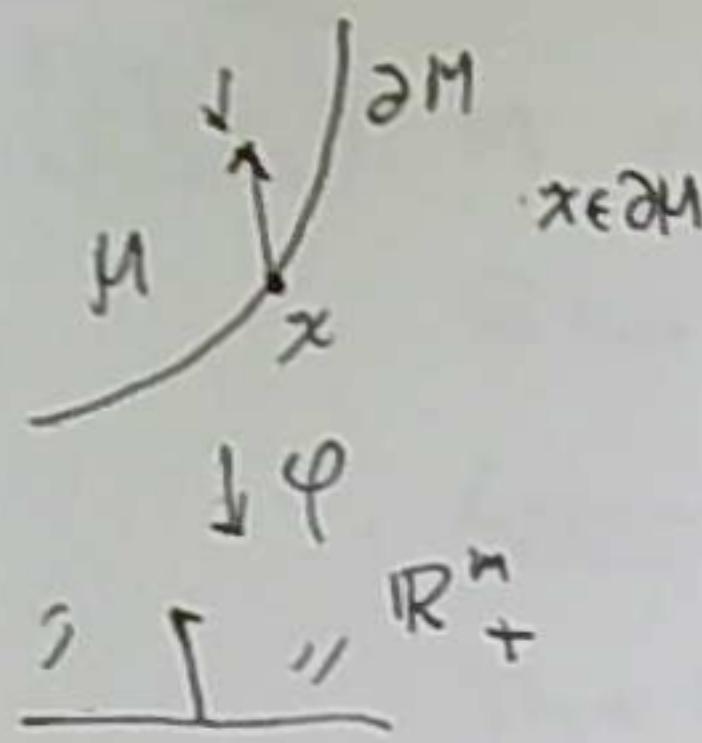
- $\varphi: U \rightarrow \mathbb{R}^n$  w/  $U \cap V$  not connected.

charts of  $M$   $\psi: V \rightarrow \mathbb{R}^n$  and  $\psi \circ \varphi^{-1}$  not positive on all of  $U \cap V$ .

Then  $M$  is not orientable

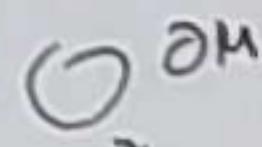


Ex. 3 Orientation induced on  $\partial M$  by an orientation on  $M$ .



$T_x M$  has 3 types of vectors  $\left\{ \begin{array}{l} T_x(\partial M) \\ \text{inward-pointing} \\ \text{outward-pointing} \end{array} \right.$

$O_x^M$  or. on  $M$



$(v_1, \dots, v_{m-1})$  positive  $\stackrel{\text{def}}{\iff}$

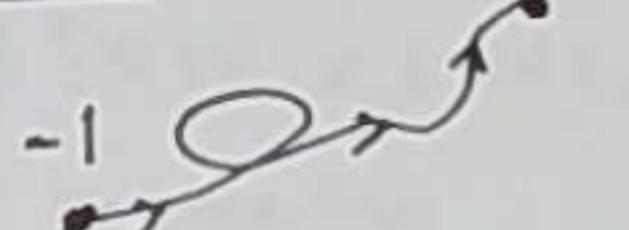
$(v, v_1, \dots, v_{m-1}) \in O$

for any outward pointing  $v$

$$\underline{\text{Ex. } S^m = \partial D^m}$$

$$\dim M = 1$$

+1



Ex. 4

$\alpha: S^m \rightarrow S^m$  antipodal map.

$$x \in S^m \quad E_x = T_x S^m \hookrightarrow \mathbb{R}^{m+1} \text{ (subspace)} \quad \xrightarrow{\text{ }} M_{m+1}.$$

$(v_1, \dots, v_m)$  positive basis of  $E_x \iff \det [x | v_1 | \dots | v_m] > 0$

$$E_{-x} = T_{-x} S = E_x$$

$$O_{-x} = -O_x \quad d\alpha(x) = -Id_{E_x} \quad \det(d\alpha(x)) = (-1)^m$$

for  $\alpha$  to be or. preserv.,

need  $d\alpha: O_x \rightarrow -O_x$  : need  $(-1)^m < 0$ , or m odd

m even :  $\alpha$  reverses orientation.

Ex. 5  $P^m = S^m / \{\text{II}, \alpha\}$

$$\alpha^2 = Id_{S^m}$$

Q) orientable?

$$\boxed{\pi \circ \alpha = \pi}$$

$$\pi: S^m \rightarrow P^m$$

m odd define an orientation on  $T_{\pi(x)} P^m$  by requiring  $d\pi(x)$  to be positive at  $x$

$$d\pi(\alpha(x)) \circ d\alpha(x) = d\pi(x).$$

$$d\alpha(x) = d\pi(x) \circ d\pi(\alpha(x))^{-1} : T_{\pi(x)} S^m \xrightarrow{\text{ }} \text{positive} \iff \alpha \text{ or. preserv.} \iff m \text{ odd}$$

n even:  $d\alpha(x)$  or. reversing

Generalization

Thm. 1  $M$  connected  $C^k$  manifold.

$G \hookrightarrow M$  by  $C^k$  diffeos, prop. discontinuously.

Assume  $M/G$  is Hausdorff.

Then  $M/G$  has a (unique)  $C^k$  manifold structure.

- If  $M$  is oriented and  $G$  acts by or.-preserving diffeos, then  $M/G$  proof: see p. 5 ( $M/G$  or.  $\Rightarrow G$  acts by or.-pres. diffeos.) is orientable.

Ex. 4

$$f(x, y) = (x, y+1) \quad g(x, y) = (x+1, 1-y)$$

$f, g \in \text{Diff}(\mathbb{R}^2)$  (i.e.:  $f$  positive,  $g$  negative)

$$gf(x, y) = g(x, y+1) = (x+1, -y)$$

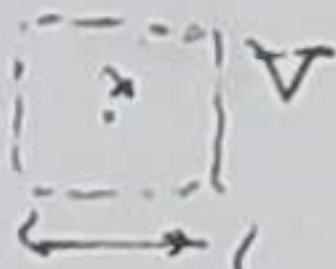
$$f^{-1}g(x, y) = f^{-1}(x+1, 1-y) = (x+1, -y) > gf = f^{-1}g.$$

any element of  $\langle f, g \rangle = G$  has the form  $f^m g^n$

$$\begin{aligned} m, n \in \mathbb{Z} \quad f^m g^n(x, y) &= \begin{cases} (x+n, y+m) & \text{if } n \text{ even} \\ (x+n, m+1-y) & \text{if } n \text{ odd} \end{cases} \\ \text{diffeos} \end{aligned}$$

$$\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$$

prop. disc.



$$g(V) \cap V = \emptyset \quad (g \neq \text{id.})$$

$$K = \mathbb{R}^2/G \quad 2\text{-dim'l non-orientable mfld}$$

$$\pi: \mathbb{R}^2 \rightarrow \mathbb{R}^2/G \text{ reg. cony. so } \pi_*(K) \approx G = \langle f, g \mid gf = f^{-1}g \rangle$$

oriented double cover

Def.  $p: \tilde{M} \rightarrow M$  (i) 2-sheeted covering (differentiable)  
(ii)  $\tilde{M}$  oriented.

(iii) If  $p(x_1) = p(x_2) \Rightarrow d\tilde{p}(x_2)^{-1} d\tilde{p}(x_1) : T_{x_1} \tilde{M} \rightarrow T_{x_2} \tilde{M}$   
is negative

(so if  $\tilde{M}$  is connected, then  $M$  is not orientable).

**Thm 2**

Any  $M$  (connected) has an oriented double covering  $\tilde{M}$ ;  
 $\tilde{M}$  is connected  $\Leftrightarrow M$  is not orientable.

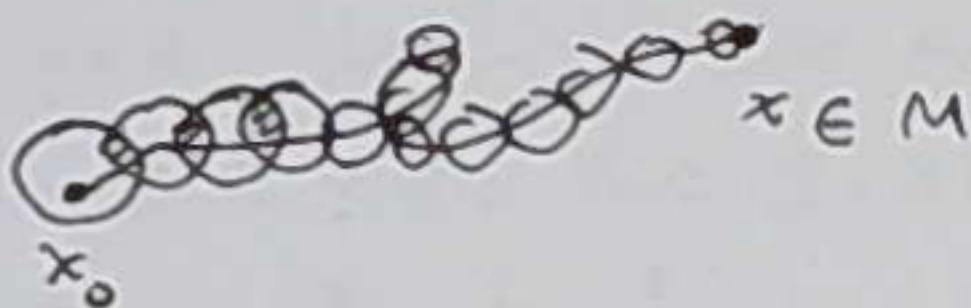
Cor.  $M$  simply connected  $\rightarrow M$  orientable.

(Any 2-sheeted cover of  $M$  w/ disconnected).

Pf (contn)

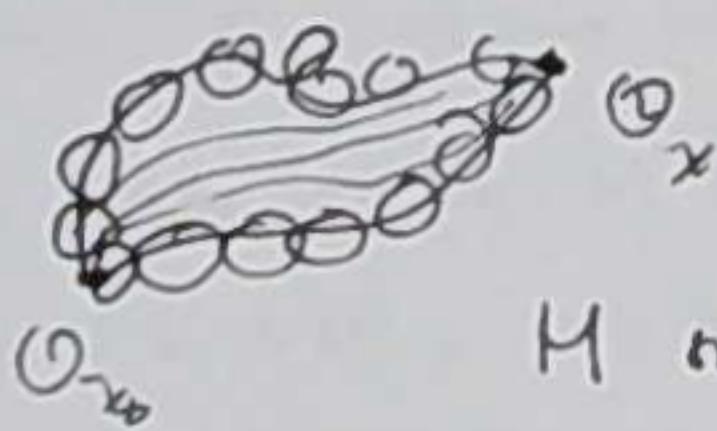
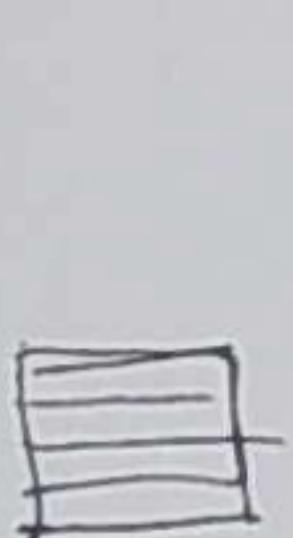
As a set  $\tilde{M} = \{(x, \mathcal{O}_x); x \in M, \mathcal{O}_x : \text{orientation of } T_x M\}$   
 $f(x, \mathcal{O}_x)_{\text{new}}$   
 $f(x, \mathcal{O}_x) = x$ .

(more details on p. 5)



Extend  $\mathcal{O}_{x_0}$  to an or. on  $x$  continuing along the path

$(\sigma(t), \mathcal{O}_{\sigma(t)}) = \tilde{\sigma}(t) : \text{lift of } \sigma \text{ to } \tilde{M}$ .



$M$  simply connected,  $\mathcal{O}_x$  does not depend on the curve.

Ex. 5 Möbius strip

$$A = (0, 5) \times (0, 1) \subset \mathbb{R}^2$$

$A_{ij} = (i, j) \times (0, 1)$  where  $0 \leq i < j \leq 5$  integers.

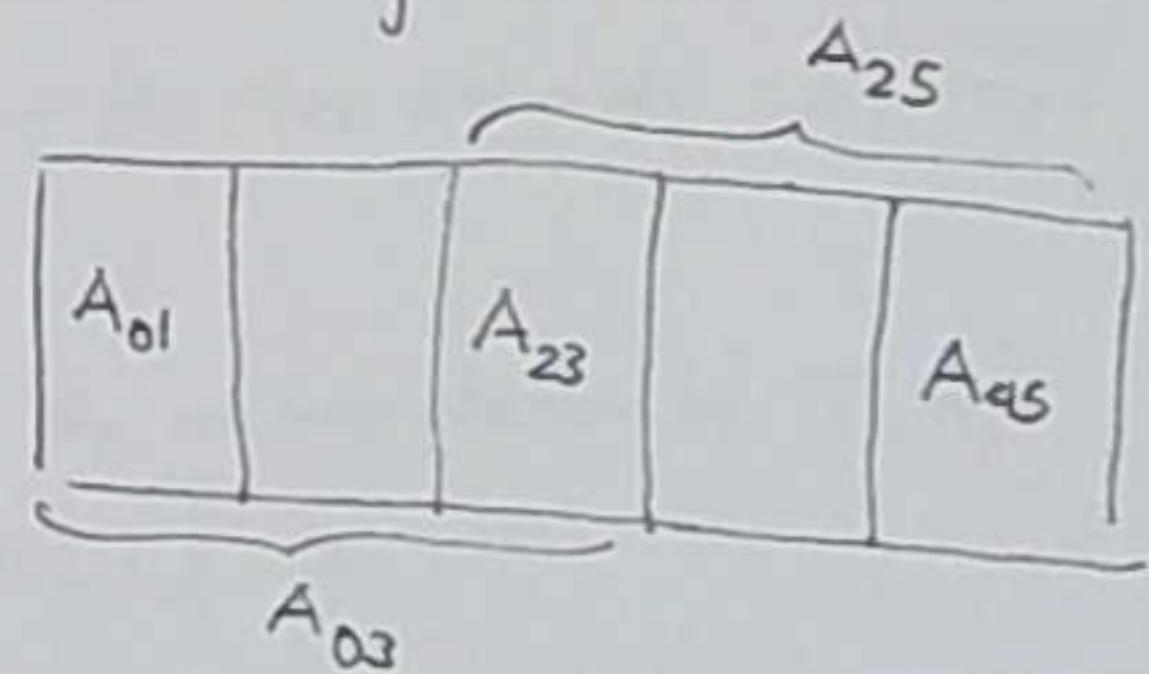
eq. relation  $(s, t) \in A_{01} \sim (s+4, 1-t) \in A_{45}$

$M = A/\sim \quad \pi: A \rightarrow M$  quotient map

$\pi|_{A_{03}}, \pi|_{A_{25}}: \text{homeo, w/ inverses } \varphi: U \rightarrow A_{03}, \psi: V \rightarrow A_{25} \text{ (charts)}$

$\varphi: U \cap V \rightarrow A_{01} \sqcup A_{23} \quad \psi: U \cap V \rightarrow A_{23} \sqcup A_{45} \quad U, V \subset M \text{ open}$

$\psi \circ \varphi^{-1}: A_{01} \sqcup A_{23} \rightarrow A_{23} \sqcup A_{45}$  is the identity in  $A_{23}$ ,  
and on  $A_{01}$  given by:  $(s, t) \mapsto (s+4, 1-t) \in A_{45}$  (orientation-reversing)  
Thus  $M$  is not orientable. (Criterion at the end of p. 0).



Proof of theorem Cover  $M$  by open sets mapped disjointly under  $\pi$ .

If  $U$  is such a set,  $\pi|_U$  is a homeom. to an open set  $U_0 \subset M/G$ .

Its inverse  $\Phi(\pi|_U)^{-1}: U_0 \rightarrow U$  is a "chart" for  $M/G$ .

(i) The 'coordinate changes' are of class  $C^k$ .

Let  $U, V$  open disjointly mapped,  $\pi|_U: U \xrightarrow{\sim} U_0$ ,  $\pi|_V: V \xrightarrow{\sim} V_0$ ,  $U_0 \cap V_0 \neq \emptyset$   
(open in  $M/G$ )

Let  $A = (\pi|_U)^{-1}(U_0 \cap V_0)$ ,  $B = (\pi|_V)^{-1}(U_0 \cap V_0)$  (open in  $M$ )

We claim  $\xi = (\pi|_V)^{-1} \circ (\pi|_U): A \rightarrow B$  is of class  $C^k$ .

Proof If  $x \in A$ ,  $\xi(x) = y \in B \Rightarrow \pi(x) = \pi(y) \Rightarrow y = \alpha(x)$  for some  $\alpha \in G$ .

Thus  $(\forall x \in A) (\exists \alpha \in G)$  s.t.  $\xi(x) = \alpha(x)$ .

Fix  $x \in A$ . Let  $z \in Z \subset A$  ( $Z$  open) s.t.  $\xi(z) \subset \alpha(A)$ . We know if  $\beta \neq \alpha$ ,  $\alpha(A) \cap \beta(A) = \emptyset$ , thus  $\xi(Z) \cap \beta(Z) = \emptyset$  if  $\beta \neq \alpha \in G$ . Thus  $\xi|_Z = \alpha|_Z$ : in a nbhd of  $Z$ , the "coord. change"  $\xi$  is given by an element of  $G$ , and hence is of class  $C^k$ . If  $\psi = (\pi|_V)^{-1}: V_0 \rightarrow V$ , we showed  $\xi = \psi_0 \psi^{-1}: \psi(U_0 \cap V_0) \rightarrow \psi(U_0 \cap V_0)$  is  $C^k$ , proving the claim  $\square$ .

Since  $\pi: M \rightarrow M/G$  is a local diff., this  $C^k$  structure on  $M/G$  is unique.

(ii) Now assume  $M$  is oriented and each  $\alpha \in G$  is a positive diff. Then if

$\pi(x) = \pi(y)$ ,  $y = \alpha(x)$  for some  $\alpha \in G$ ; with  $d\pi(y) \circ d\alpha(x) = d\pi(x)$ , or

$d\pi(y)^{-1} \circ d\pi(x) = d\alpha(x)$  (positive). Thus  $\pi$  induces an orientation on  $M/G$ .  
Given  $\alpha \in G$        $(y = \alpha(x))$

Conversely if  $M/G$  is or'ble,  $d\pi(y)^{-1} \circ d\pi(x)$  is positive whenever  $\pi(x) = \pi(y)$ . Hence  $d\alpha$  is positive.

Proof of theorem (outline). Let  $\tilde{M} = \{(x, \mathcal{O}_x); \mathcal{O}_x \text{ an orientation of } T_x M\}$

$p(x, \mathcal{O}_x) = x \in M$ ,  $p^{-1}(x) = \{(x, \mathcal{O}_x), (x, -\mathcal{O}_x)\}$ .  $\star$

$C^k$  structure on  $\tilde{M}$ : for any oriented open set  $U \subset M$ , let  $\tilde{U}$  be the set of pairs  $(x, \mathcal{O}_x)$ , where  $x \in U$  and  $\mathcal{O}_x$  is the orientation in  $U$ .  $\varphi_U = p|_{\tilde{U}}: \tilde{U} \rightarrow U$  is bijective. Given  $\varphi_U$  and  $\varphi_V$  with  $\tilde{U} \cap \tilde{V} \neq \emptyset$ , the "coordinate change"  $\varphi_V \circ \varphi_U^{-1}$  is just the identity (sme  $\varphi_U|_{\tilde{U} \cap \tilde{V}} = \varphi_V|_{\tilde{U} \cap \tilde{V}}$ ). Thus the charts  $\varphi_U$  determine

on  $\tilde{M}$  a  $C^k$  structure, coming from the one on  $M$  (so  $p: \tilde{M} \rightarrow M$  is a local diff.)

$\tilde{M}$  has a natural orientation, co-induced by  $p$  (i.e. require  $d\varphi_U(\tilde{x})$  to be positive  
 $\forall \tilde{x}$ )