Prop. disc. group actions (by homeomorphisms)

$G \curvearrowright X$ by homeo. prop. disc: each $x \in X$ has a disjointly mapped nbd $U \subset X$

last time
$$\begin{array}{c} X \\ \pi \downarrow \\ X/G \text{ (quotient top.)} \end{array}$$
is a covering map.

Rk If X is a (diff'bk) mfld and G acts by diffeos., then X/G is loc. euclidean and 2nd ctble (if X is) but not nec. Hausdorff.

Exercise X/G is Hausdorff if $\Gamma = \{(x, gx);\ g \in G,\ x \in X\}$ is closed in $X \times X$ (e.g. if G acts by isometries).

$p: \tilde{X} \xrightarrow[\text{covering}]{} X$, $\tilde{X}$ connected $\quad G = \mathrm{Aut}(\tilde{X}|X) \quad (p \circ f = p)$

Prop: G acts on $\tilde{X}$ prop. discontinuously.

Pf: Given $\tilde{x} \in \tilde{X}$, let U be an evenly covered nbd of $x = p(\tilde{x})$.

Let V: nbd of $\tilde{x}$ s.t. $p|_V : V \xrightarrow{\sim} U$ is a homeo.

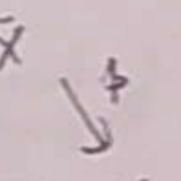
Let $f \in G$, $f \neq id_{\tilde{X}}$. Then if $v \in V$ $f(v) \notin V$ (since $f \neq id_{\tilde{X}}$).

$f(v) \notin V$ since v and $f(v)$ map to the same pt (under p)

Then $V \cap f(V) = \emptyset$: V is mapped disjointly under G.

Thm (Classf'n of reg. covers). X conn., loc. path conn.

If $p: \bar{X} \to X$ is regular covering then $\exists$ a homeo $h: \tilde{X}/G \to X$ with $h \circ \pi = p$, $G = \mathrm{Aut}(\tilde{X}|X)$



Pf: Let $\tilde{x}, \tilde{y} \in \tilde{X}$.

$p(\tilde{x}) = p(\tilde{y}) \iff \exists f \in G$ s.t. $f(\tilde{x}) = \tilde{y} \iff G\tilde{x} = G\tilde{y} \iff \pi(\tilde{x}) = \pi(\tilde{y})$

This defines a bijection h from $\tilde{X}/G$ to X s.t. $h \circ \pi = p$.

$V \subset X$ open $\to$ $h^{-1}(V) = \pi(p^{-1}(V))$. p cont. π open $\to$ h cont.

$U \subset \tilde{X}/G$ open $\to$ $h(U) = p(\pi^{-1}(U))$ π cont., p open $\to$ h open.
$\subset X$ open $\Rightarrow$ h is a homeo.

Existence of simply connected covers. ("universal covering space")

necessary condition

$p: \tilde{X} \to X$ $\tilde{X}$ simply-conn., $\tilde{x}_0 \in p^{-1}(x_0)$.

Then $\exists\, U \subset X$ nbd of x_0 s.t. $i_* \pi_1(U, x_0) \to \pi_1(X, x_0)$ is trivial

Pf. Let U be evenly covered by p; $p^{-1}(U) = \coprod_{\alpha \in A} U_\alpha$ $U_\alpha \subset \tilde{X}$

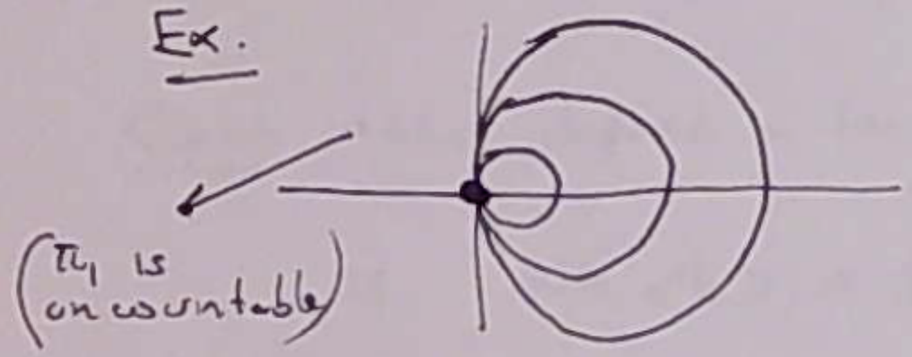
$a \in \Omega_{x_0}(U) \to \exists\, \tilde{a} \in \Omega_{\tilde{x}_0}(U_\alpha)$ (since $p|_{U_\alpha}: U_\alpha \to U$ is homeo.)

$\forall$ s.c. $\tilde{a} \overset{H}{\simeq} e_{\tilde{x}_0}$ (in $\tilde{X}$) $\quad p \circ H$ is a homotopy in X from a to e_{x_0}

Def. X is semilocally simply connected if $\forall x \in X$ $\exists\, U \subset X$ nbd of x s.t. $i_*: \pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Rk. Ex If X is loc. simply connected (i.e. $\forall x$ $\exists\, U \subset X$ nbd of x simply-conn.)
in part. X manifold

Ex.



(see [Munkres]).

$X = \bigcup C_n$ C_n i circle w/ radius $1/n$ tgt to y-axis at 0
(w/ topology induced from $\mathbb{R}^2$).

U: nbd of 0 in X.

$i_* \pi_1(U, 0) \to \pi_1(X, 0)$.

Rk X is not homeo. to a ctble wedge $\bigvee_{n \geq 1} C_n$. ("coherent topology")

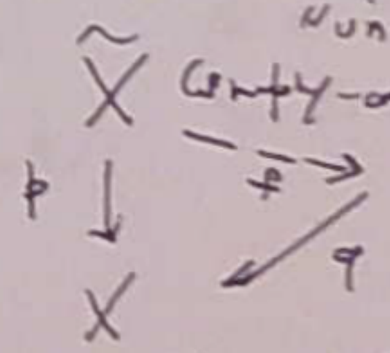
Thm. X loc. path conn, connected, semi-locally simply-conn $\Rightarrow$ X has a simply-conn. cover. "universal cover"

Rk Let $\tilde{Y} \xrightarrow{q} X$ be a covg ($\tilde{Y}$ connected).

Then: $f: \tilde{X} \to Y$ (covg) s.t. $q \circ f = p$

since $\forall \tilde{x} \in \tilde{X}$ and $\tilde{y} \in \tilde{Y}$ w/ $p(\tilde{x}) = q(\tilde{y})$.

we have $\{0\} = p_* \pi_1(\tilde{X}, x) \subset q_* \pi_1(\tilde{Y}, \tilde{y}) \to$ a covg hom. $f: \tilde{X} \to \tilde{Y}$ exists. ($\Rightarrow f$ is a covering)
(hence)

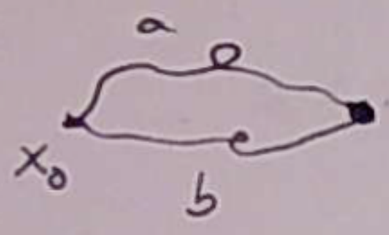


Pf. (outline)

① $\tilde{X}$ as a set. Fix $x_0 \in X$

$\mathcal{P}_{x_0} = \{$ paths in X starting at $x_0\}$

eq. rel'n in $\mathcal{P}_{x_0}$: $a \equiv b$ if they have the same endpt $a(1) = b(1)$ and are path-htopic in X



$\tilde{X} = \mathcal{P}_{x_0}/\equiv$ $\quad p: \tilde{X} \longrightarrow X$

$p(\langle a\rangle) = a(1)$.

(p is onto since X is path-conn.).

② topology on $\tilde{X}$

basis of nbds of $\langle a\rangle \in \tilde{X}$ $\quad \mathcal{U} = \{ U \subset X$ open s.t. $\forall$ loop γ in U, $\gamma \simeq$ const. $x_1 \}$

$B(U, \langle a\rangle) = \{ \langle a * b\rangle ;\ b(I) \subset U,\ b(0) = a(1) \}$

Claim this defines a loc. basis (at $\langle a\rangle$) for a top. on $\tilde{X}$.

if $\langle b\rangle \in B(U, \langle a\rangle)$ then $B(U, \langle a\rangle) = B(U, \langle b\rangle)$

③ p is cont., open and $p|_{B(U,\langle a\rangle)}$ is bijective onto U.

④ p is a cov. map.

Let $U \subset \mathcal{U}$. Then U is evenly covered by p.

$$p^{-1}(U) = \coprod_{a \in \mathcal{P}_{x_0 x}} B(U, \langle a\rangle) \qquad (x \in U \text{ fixed})$$

(disjoint).

⑤ $\tilde{X}$ connected. $\tilde{x}_0 = \langle e_{x_0} \rangle$ const path at x_0.

$\langle a \rangle \in \tilde{X}$ $\quad a \in \mathcal{P}_{x_0}$. $\quad a : I \longrightarrow X$ path

Let $\quad a_t(s) = a(ts) \quad t \in [0,1] \; s \in [0,1]$

$a_t(1) = a(t)$.

Let $\quad \tilde{a} : I \longrightarrow \tilde{X}$

$\tilde{a}(t) = \langle a_t \rangle \quad \tilde{a}$ <u>lifts</u> a to $\tilde{X}$ (from $\tilde{x}_0$)

$\tilde{a}(1) = \langle a_1 \rangle = \langle a \rangle$

⑥ X simply-connected.

$[a]_X \in p_* \pi_1(\tilde{X}, \tilde{x}_0) \Leftrightarrow \tilde{a}$ is a loop at $\tilde{x}_0$ $\quad (\tilde{a}(1) = \tilde{a}(0))$

$\Leftrightarrow \quad \langle a \rangle = \langle a_1 \rangle = \langle e_{x_0} \rangle$

$a_1 \equiv e_{x_0} \Leftrightarrow a \simeq$ const in X.

$a \sim e_{x_0}$

$\Leftrightarrow \quad p_* : \pi_1(X, x_0) \longrightarrow \{e_0\}$ (but recall p_* is injective)

<u>Remark</u>: more generally:

<u>Thm</u> Let X be loc. path connected, connected, semi-locally simply connected.

Fix $x_0 \in X$ and let $H \subset \pi_1(X, x_0)$ be <u>any</u> subgroup. Then

$\exists$ a covering $p : \tilde{X} \longrightarrow X$, w/ $\tilde{X}$ connected, and $\tilde{x}_0 \in p^{-1}(x_0)$,

s.t. $\quad p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$