

# Group actions (right actions, left actions)

① Right action of a group  $G$  on a set  $S$

$\Phi: G \times S \rightarrow S$  s.t.  $\Phi_g(\cdot) = \Phi(g, \cdot)$  is a bijection of  $S$   
denoted  $\text{fix}_g (g, x) \mapsto x g$   $\Phi_e = \text{id}_S$

$$x(g h) = (x g) h$$

- orbit of  $x \in S$   $xG = \{xg \mid g \in G\}$
- $G$  acts transitively on  $S$  if one orbit (and hence any) equals  $S$ .  
 $\forall x, y \in S \exists g \in G$  s.t.  $y = x g$ .
- isotropy group of  $x \in S$   $I_x = \{g \in G \mid xg = x\} I_x \subset G$  subgroup.
- If  $\boxed{y = xh}$   
 then  $y g = y \Leftrightarrow x(h g h^{-1}) = x$   
 $\uparrow$   
 $x h g = x h \Leftrightarrow I_x = h I_y h^{-1}$

so if the action is transitive, all isotropy groups are conjugate.

- suppose  $S \hookrightarrow G$  is transitive; fix  $x_0 \in S$ .

Then  $\varphi: G \rightarrow S \quad \varphi(g) = x_0 g$  is onto  $S$ .

$$\varphi(g) = \varphi(h) \Leftrightarrow x_0 g = x_0 h \Leftrightarrow hg^{-1} \in I_{x_0}$$

$\bar{\varphi}: G/I_{x_0} \rightarrow S$  is a bijection

Prop-  $\phi: X \rightarrow X$  conng ( $\tilde{X}, X$  conn; loc path conn.)

$\forall x \in X \quad \pi_1(X, x)$  acts transitively (on the right) on  $\phi^{-1}(x)$  with isotropy gp  
 $(\text{of } \tilde{x} \in \phi^{-1}(x))$

$$H(\tilde{x}) = \phi_* \pi_1(\tilde{X}, \tilde{x}) \subset \pi_1(X, x).$$

Pf. Given  $\alpha \in \pi_1(X, x)$ ,  $\tilde{x} \in \phi^{-1}(x)$ , define  $\tilde{x}\alpha$  as follows:

take  $a \in \alpha$  (loop at  $x$ ). Lift  $a$  from  $\tilde{x}$  to get  $\tilde{a}: I \rightarrow \tilde{X}$ ,

let  $\tilde{x}\alpha = \tilde{a}(1)$ . (Well-defined).

(3)

$\tilde{x}\alpha = \tilde{x} \iff \tilde{\alpha}$  is a loop at  $\tilde{x} \iff \alpha \in H(\tilde{x})$  (so the isotropy gr. of  $\tilde{x}$  is  $H(\tilde{x})$ )

Given  $\tilde{x}, \tilde{y} \in \phi^{-1}(x)$ , let  $\tilde{\alpha}$  be a path in  $\tilde{X}$  from  $\tilde{x}$  to  $\tilde{y}$ .

Then  $a = \phi \circ \tilde{\alpha} \in \Omega_x(X)$ . Let  $\alpha \in [a]$ . Then  $\tilde{x}\alpha = \tilde{y}$ .

Cor. 1. ( $\tilde{X}$  path connected). Then the no. of sheets of the cover equals

the index of  $H(\tilde{x})$  in  $\pi_1(X, x)$  ( $\forall x \in X, \forall \tilde{x} \in \phi^{-1}(x)$ ).

Cor. 2 Any 2-sheeted cover  $\phi: \tilde{X} \rightarrow X$  is regular  
(any subgroup of index 2 is normal).

Cor. 3  $\tilde{X}$  simply-connected  $\Rightarrow$  # of sheets =  $\# \pi_1(X, x)$ .

Ex.  $P^n = S^n / \{id, \alpha\}$  (manifold)  $S^n \rightarrow P^n$  2-sheeted  
( $n \geq 2$ )  $\# \pi_1(P^n) = 2 \Rightarrow \pi_1(P^n) = \mathbb{Z}_2$  covering map.

(2) Left action of  $\text{Aut}(\tilde{X}|X)$  on  $\tilde{X}$ . ( $\phi: \tilde{X} \rightarrow X$   $\tilde{X}$  conn.)

recall fund. lifting thm. implies  $(\tilde{x}_0, \tilde{x}_1 \in \phi^{-1}(x_0))$

$\tilde{x} \xrightarrow{\text{f}} \tilde{X} \xrightarrow{\phi} X$   $\exists f \in \text{Hom}(\tilde{X}|X) \quad f(\tilde{x}_0) = \tilde{x}_1 \iff H(\tilde{x}_0) \subset H(\tilde{x}_1).$  (\*)  
 $\exists f \in \text{Aut}(\tilde{X}|X) \quad f(\tilde{x}_0) = \tilde{x}_1 \iff H(\tilde{x}_0) = H(\tilde{x}_1).$

(Def.)  $f \in \text{Aut}(\tilde{X}|X)$  means  $f: \tilde{X} \rightarrow \tilde{X}$  homeo. s.t.  $\phi \circ \tilde{f} = f$ ,  
so  $f$  takes fibers of  $\phi$  to themselves bijectively)

→ group action of  $\overbrace{\text{Aut}(\tilde{X}|X)}$  on  $\tilde{X}$  by homeomorphisms. (acting "on the left"  
(group str. given by composition).  $f \cdot \tilde{x} = \tilde{f}(\tilde{x})$ )

(Q) Transitive? What are the isotropy grps?

Recall If  $\tilde{X} \xrightarrow{\phi} X$  is regular all groups  $H(\tilde{x}_i)$   $\tilde{x}_i \in \phi^{-1}(x)$ .

are normal (and equal) so.  $\text{Aut}(\tilde{X}|X)$  acts transitively on each fiber.

(from \*)

(3)

Conversely if  $G = \text{Aut}(\tilde{X}|X)$  acts transitively on fibers of  $\tilde{\rho}$ , then  $\tilde{\rho}$  is regular.

Pf If  $a$  is a loop and lifts to  $\tilde{a}$  from  $\tilde{x} \in \tilde{\rho}^{-1}(x)$ ,  $\tilde{\rho}^{-1}(x) = \{f(\tilde{x}) : f \in G\}$ , so  $f \circ \tilde{a}$  is the lift of  $a$  from  $f(\tilde{x})$ : either all loops are closed or all are open.  $\square$

recall if  $\tilde{x}_0 \in \tilde{\rho}^{-1}(x_0)$ , any other pt on the fiber  $\tilde{\rho}^{-1}(x_0)$  has the form  $\tilde{x}_0 \cdot \alpha$ ,  $\alpha \in \pi_1(X, x_0)$ .

$\Rightarrow \exists f \in \text{Aut}(\tilde{X}|X)$  s.t.  $f(\tilde{x}_0) = \underbrace{\tilde{x}_0}_{\tilde{x}} \cdot \alpha \Leftrightarrow H(\tilde{x}) = H(\tilde{x}_0) = \alpha^{-1} H(\tilde{x}_0) \alpha$   
 $(\tilde{x} \cdot \alpha = \alpha \tilde{x}, \tilde{a} : \text{the lift from } \tilde{x}_0 \text{ of a loop representing } \alpha)$  i.e.  $\Leftrightarrow \alpha \in N(H(\tilde{x}_0))$

Def. : normalizer of a subgroup  $H \subset G$ .

$$N(H) = \{g \in G \mid g^{-1}Hg = H\} \quad (\text{subgroup of } G \text{ containing } H)$$

note  $N(H) = G \Leftrightarrow H \triangleleft G$ .  $H \triangleleft N(H)$  normal.

Thm  $\forall \tilde{x} \in \tilde{X}, \text{Aut}(\tilde{X}|X) \cong N(H(\tilde{x})) / H(\tilde{x})$  (isom. of groups)

Proof We define  $\varphi: N(H(\tilde{x})) \rightarrow \text{Aut}(\tilde{X}|X)$

$(N(H(\tilde{x})) \subset \pi_1(X, \tilde{x})$  subgroup)

$\alpha \mapsto f$   
 where  $f(\tilde{x}) = \tilde{x} \cdot \alpha$  (this uniquely defines  $f$ )

Claims  $(f \text{ exists since } \alpha \in N(H(\tilde{x})))$  [see (\*\*)] \*

(1)  $\varphi$  is a group homomorph.

(2)  $\ker \varphi = H(\tilde{x})$ .

(3)  $\varphi$  is surjective.

(\*  $f$  is uniquely det by using its value on a single  $\tilde{x} \in \tilde{X}$ , by uniqueness in the lifting theorem.)

(4)

Proof of thm

we define  $\varphi: N(H(\tilde{x}_0)) \rightarrow \text{Aut}(\tilde{X}|X)$

$$\alpha \mapsto f$$

where  $f(\tilde{x}) = \tilde{x}_0 \cdot \alpha$  (this uniquely defines  $f$ )

(Note  $f$  exists since  $\alpha \in N(H(\tilde{x}_0))$ )

Claims (1)  $\varphi$  is a group homomorphism

(2)  $\ker \varphi = H(\tilde{x}_0)$

(3)  $\varphi$  is surjective

For (1) we need the lemma:

Lemma If  $f \in \text{Aut}(\tilde{X}|X)$ ,  $\tilde{x} \in \tilde{X}$  and  $\alpha \in \pi_1(X, p(\tilde{x}))$ ,

we have:  $f(\tilde{x} \cdot \alpha) = f(\tilde{x}) \cdot \alpha$

("the left action of Aut and the right action of  $\pi_1$  on the fibers commute")

Pf. Let  $\alpha = [\alpha]$ ,  $\tilde{\alpha}$  the lift of  $\alpha$  from  $\tilde{X}$ .

Then  $f \circ \tilde{\alpha}$  is the lift of  $\alpha$  from  $f(\tilde{x})$ .

Thus  $\tilde{x} \cdot \alpha = \tilde{\alpha}(1)$ ,  $f(\tilde{x}) \cdot \alpha = (f \circ \tilde{\alpha})(1)$ .

But  $f(\tilde{\alpha}(1)) = (f \circ \tilde{\alpha})(1)$ .

Pf of (1)

Let  $\varphi(\alpha) = f$ ,  $\varphi(\beta) = g$

i.e.  $f(\tilde{x}_0) = \tilde{x}_0 \cdot \alpha$ ,  $g(\tilde{x}_0) = \tilde{x}_0 \cdot \beta$

$(f \circ g)(\tilde{x}_0) = f(g(\tilde{x}_0)) = f(\tilde{x}_0 \cdot \beta)$

$\stackrel{\text{lem}}{=} f(\tilde{x}_0) \cdot \beta = (\tilde{x}_0 \cdot \alpha) \cdot \beta = \tilde{x}_0 (\alpha \beta)$

i.e.  $\varphi(f \circ g) = \alpha \beta = \varphi(f) \varphi(g)$

Pf of (2)

$\nexists \tilde{x}_0 \cdot \alpha = \tilde{x}_0 \Leftrightarrow \alpha \in H(\tilde{x}_0)$  ( $\varphi(\alpha) = f \circ id_X \Leftrightarrow \alpha \in H(\tilde{x}_0)$ )

Pf of (3)

Let  $f \in \text{Aut}(\tilde{X}|X)$ ,  $f(\tilde{x}_0) = \tilde{x}_1 \in P_{\tilde{x}_0}^{-1}(x_0)$

Then  $\tilde{x}_1 = \tilde{x}_0 \cdot \alpha$ , and since  $f \in \text{Aut}$ ,  $\alpha \in N(H(\tilde{x}_0))$ .

Thus ~~some~~  $\alpha \in N(H(\tilde{x}_0))$ .

$\varphi(\alpha) = f$  for some  $\alpha \in N(H(\tilde{x}_0))$ .

Corollaries ( $\tilde{X}$ : connected, loc. path connected)

$$\text{Cor.1} \quad \tilde{X} \xrightarrow{p} X \text{ regular} \Rightarrow \text{Aut}(\tilde{X}|X) \approx \pi_1(X, x_0) / H(\tilde{x}_0) \quad \forall \tilde{x}_0 \in p^{-1}(x) \\ (\text{group isom.})$$

$$\text{Cor.2} \quad \tilde{X} \text{ simply connected} \Rightarrow \text{Aut}(\tilde{X}|X) \approx \pi_1(X, x_0) \\ (\text{since } H(\tilde{x}_0) = \{e_{x_0}\})$$

Thus in this case  $\pi_1(X, x_0)$  acts on  $\tilde{X}$  by homeomorphisms (commuting w/p)

description of the action

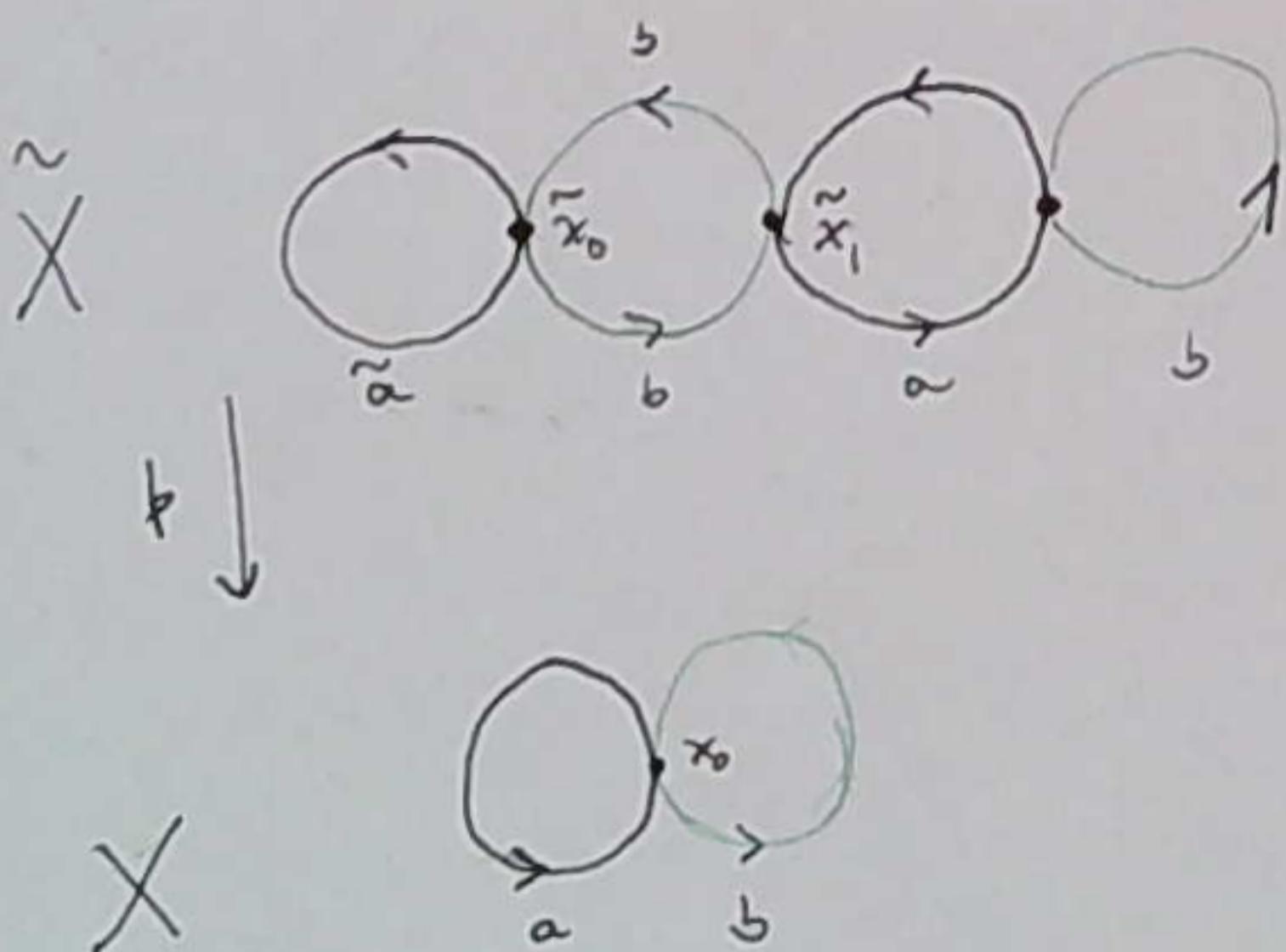
Given  $\alpha \in \pi_1(X, x_0)$ , the associated  $f \in \text{Aut}(\tilde{X}|X)$  acts as follows.

Let  $\tilde{x} \in \tilde{X}$ . Join  $\tilde{x}$  to  $\tilde{x}_0 \in p^{-1}(x)$  by a path  $\tilde{b}$  ( $\tilde{b}: \mathbb{I} \rightarrow \tilde{X}$   
 $\tilde{b}(0) = \tilde{x}$ )  
some

Let  $b = p \circ \tilde{b}$ ,  $a \in \alpha$  (a loop at  $x_0$ ),  $x = p(\tilde{x})$ .

Then  $bab^{-1} \in \Omega_x(x)$ . Lift  $bab^{-1}$  from  $\tilde{x}$  to get a path ending  
at  $\tilde{y} \in p^{-1}(x)$ . Then  $f(\tilde{x}) = \tilde{y}$ .

Example Non-regular 3-fold covering of  $S^1 \vee S^1$



$$\nexists f: \tilde{X} \rightarrow \tilde{X} \quad (\quad p \circ f = p \\ \text{s.t. } f(\tilde{x}_0) = f(\tilde{x}_1)$$

Letting  $\tilde{a}$  be the lift of  $a$  from  $\tilde{x}_0$  (closed),  
 $f \circ \tilde{a}$  would be the lift of  $a$  from  $\tilde{x}_1$ ,  
hence also closed.

But the lift of  $a$  from  $\tilde{x}_1$  is open