

last week

transversality preimage thm for manifolds w/ bdy.

- nonexistence of retractions $M \rightarrow \partial M$.
 - Brouwer fixed pt. for $D^n \subset \mathbb{R}^n$.
-) Sard's theorem.
classifn of 1-mfds w/ ∂ .

next goal : \mathbb{Z}_2 - homotopy invariants (intersection number, degree, winding number...)
 \downarrow
 \mathbb{Z} - invariants. (orientability).

existence thm

Homotopy transversality thm.

$f: X \rightarrow Y$ diff'ble map. ($\partial Y = \emptyset$) $Z \subset Y$ submanifold w/o bdy.
 $\Rightarrow \exists g: X \rightarrow Y$ diff'ble homotopic to f , $g \pitchfork Z$, $\partial g \pitchfork Z$.
 (g can be taken in an arb. $W^1(X, Y)$ nbd of f .)

2 ingredients

- parametrized transversality thm.
- tubular neighborhoods of submanifolds

Parametrized transversality thm.

$F: X \times S \rightarrow Y$ diff'ble. $\partial S = \emptyset, \partial Y = \emptyset$
 $Z \subset Y$ w/o bdy.
 $f_s: X \rightarrow Y, s \in S$

Assume $F \pitchfork Z, \partial F \pitchfork Z$ $\partial(X \times S) = (\partial X) \times S$

Then for a.e. $s \in S$ (\Rightarrow for a dense set of $s \in S$)

$f_s: X \rightarrow Y, \partial f_s: \partial X \rightarrow Y$ are transv. to Z .

Pf

Consider

Let $W = F^{-1}(Z) \subset X \times S$

$\pi_W: W \rightarrow S$: restriction of $\pi: X \times S \rightarrow S$ to W submanifold

Claim

If $s \in S$ is a reg. value of π_W then $f_s \pitchfork Z$ ($f_s: X \rightarrow Y$).
 then follows from Sard's thm.

Pf of claim Suppose $f_s(x) = z \in Z$ ($F(x,s) = z \in Z$).

know $dF(x,s) [T_{(x,s)}(X \times S)] + T_z Z = T_z Y$
(since $F \pitchfork Z$) i.e.

$(\forall a \in T_z Y) \exists b \in T_{(x,s)}(X \times S)$ s.t. $dF(x,s)[b] - a \in T_z Z$ (*)

want $\exists v \in T_x X$ s.t. $df_s(x)[v] - a \in T_z Z$
(since a is arb, this says $df_s(x)[T_x X] + T_z Z = T_z Y$)

note $T_{(x,s)}(X \times S) = (T_x X) \times (T_s S)$

so $b = (w, e)$ $w \in T_x X$ $e \in T_s S$.

$df_s(x)[w] + dF(x,s)[(0, e)] - a \in T_z Z$ $b = (w, 0) + (0, e)$

By assumption $d\pi_W(x,s) : T_{(x,s)} W \rightarrow T_s S$ is onto

so \exists a vector of the form $(u, e) \in T_{(x,s)} W$ $u \in T_x X$

Let $v = w - u \in T_x X$

$$\begin{aligned} df_s(x)[v] - a &= dF(x,s) [(w, e) - (u, e)] - a \\ &= dF(x,s) \underbrace{[(w, e)]}_{b} - a - dF(x,s) [(u, e)] \\ &\in T_z Z \quad (*) \end{aligned}$$

(so both are in $T_z Z$).

$$dF(x,s) : T_{(x,s)} W \rightarrow T_z Z$$

since $F(x,s) = z$

proves want,
hence claim.

Example $Y = \mathbb{R}^M$ $Z \subset \mathbb{R}^M$ submfld $\partial Z = \emptyset$ (3)

$f: X \rightarrow \mathbb{R}^M$ diff'ble want nearby diff'ble map

$g: X \rightarrow \mathbb{R}^M$, htoprc to f

and $g \pitchfork Z$

idea find g of the form

$$g(x) = f(x) + s \quad s \in \mathbb{R}^M \text{ w/small norm.}$$

consider

$$F(x, s) = f(x) + s, \text{ def. on } X \times S$$

For fixed $x \in X$ the map

S : small open ball in \mathbb{R}^M .

$s \mapsto f(x) + s$ is a submersion from S to \mathbb{R}^M .

In part.

$F \llcorner X: X \times S \rightarrow \mathbb{R}^M$ is also a submersion, hence \pitchfork to Z .

From the par. transversality thm: for a.e. $s \in S$

$g = f_s: X \rightarrow \mathbb{R}^M$ is transversal to Z

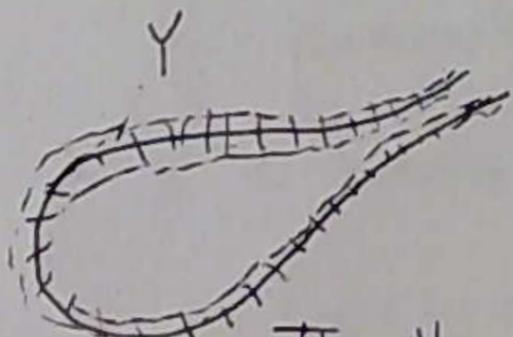
(clearly homotopic to f via $H(x, t) = f(x) + ts, t \in [0, 1]$)

Tubular neighborhood thm

$Y \subset \mathbb{R}^M$ submanifold w/o bdy. (embedded)

$\varepsilon: Y \rightarrow \mathbb{R}_+$ diff'ble.

$$Y^\varepsilon = \{ w \in \mathbb{R}^M \mid \text{for some } y \in Y, \|w - y\| < \varepsilon(y) \}$$



Then $\exists \pi: Y^\varepsilon \rightarrow Y$ diff'ble submersion,
equal to the ~~map~~ identity on Y .

("least closest pt projection")

Thm Homotopy transversality theorem.

$f: X \rightarrow Y$ diff'ble.

$Z \subset Y$ submfld w/o bdy.

$\Rightarrow \exists g: X \rightarrow Y$ homotopic to f , $g \pitchfork Z$.

Homotopy transversality theorem

(4)

Proof Let $S \subset \mathbb{R}^M$ be the open unit ball.

Recall: $\pi: Y^\varepsilon \rightarrow Y$ submersion.

Let $F: X \times S \rightarrow Y$ $\|s\| < 1$

$$F(x, s) = \pi \left(\underbrace{f(x) + \varepsilon(f(x))s}_{\in Y^\varepsilon} \right) \in Y.$$

$$F(x, 0) = f(x).$$

For fixed $x: s \mapsto f(x) + \varepsilon(f(x))s$ is submersion
 π is a submersion

so $s \mapsto F(x, s)$ is a submersion $S \rightarrow Y$

In part. $F: X \times S \rightarrow Y$ is a submersion, in part
transversal to Z .

By parameterized transversality: for a.e. $s \in S$

$$f_s: X \rightarrow Y$$

$$f_s(x) = \pi(f(x) + \varepsilon(f(x))s)$$

is transversal to Z . Fix such an s , let $g = f_s$.

g is homotopic to f via:

$$H(x, t) = \pi(f(x) + \varepsilon f(x)ts) \quad t \in [0, 1]$$

$$(\pi(f(x)) = f(x) \text{ since } f(x) \in Y)$$

Rk. g can be taken arb. $W'(X, Y)$ close to f