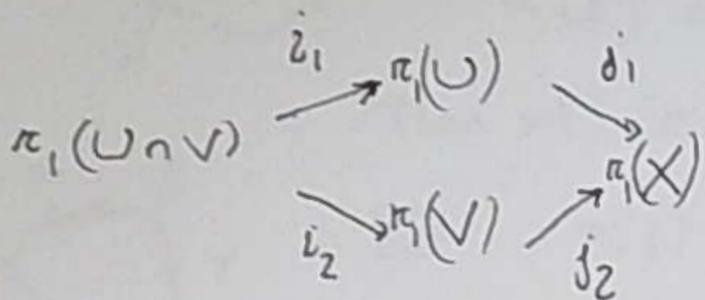


Seifert-van Kampen thm.

①

$X = U \cup V$ U, V open, path connected
 $x_0 \in U \cap V$, path-conn.



$$(j_1 \circ i_1) ([f]_{U \cup V}) = (j_2 \circ i_2) ([f]_{U \cup V}) \\ = i [f]_{U \cup V}$$

$\pi_1(U \cup V)$ introduces additional relations in $\pi_1(U) * \pi_1(V)$

~~$i_1 [f]_{U \cup V} = i_2 [f]_{U \cup V}$~~

$$i : \pi_1(U \cup V) \rightarrow \pi_1(X)$$

$\pi_1(U) \hookrightarrow \pi_1(U) * \pi_1(V)$
 $\pi_1(V) \hookrightarrow \pi_1(U) * \pi_1(V)$
inclusions (not explicit in the notation)

$$N = \text{norm} \langle i_1 [f]_{U \cup V} = (i_2 [f]_{U \cup V})^{-1}; [f] \in \pi_1(U \cup V) \rangle \\ \triangleleft \pi_1(U) * \pi_1(V).$$

Thm $\pi_1(X) = \pi_1(U) * \pi_1(V) / N$

Step 1 (geometric)

$$\Phi : \pi_1(X) \xrightarrow{=} \pi_1(U) * \pi_1(V) / N$$

$$\phi_1 : \pi_1(U) \rightarrow H$$

$$\phi_2 : \pi_1(V) \rightarrow H$$

natural (inclusion-projection)

(st.) $\left\{ \begin{array}{l} \text{if } \alpha = j_1 [f]_U \\ \text{if } \alpha = j_2 [f]_V \end{array} \right.$ then $\Phi(\alpha) = \phi_1 [f]_U$
then $\Phi(\alpha) = \phi_2 [f]_V$.

step 1.1

If $\alpha = [f]_U \in \pi_1(U)$ $f \in \Omega_{x_0}(U)$

define $p(f) = \phi_1 [f]_U$

If $f \in \Omega_{x_0}(V)$ define $p(f) = \phi_2 [f]_V$.

Then (i) $p(f)$ depends only on $[f]_U$ (or $[f]_V$).

(ii) $p(f+g) = p(f) \cdot p(g)$ $f, g \in \Omega_{x_0}(U)$.
(since ϕ_i : hom.) \rightarrow product in H

Step 2 (algebraic)

Let $\psi: \pi_1(U) * \pi_1(V) \xrightarrow{\text{hom.}} \pi_1(X)$ extending (existence from group theory)

$$j_1: \pi_1(U) \rightarrow \pi_1(X), \quad j_2: \pi_1(V) \rightarrow \pi_1(X)$$

$$\text{z.e. } \begin{cases} \psi([a]_U) = j_1[a]_U & [a]_U \in \pi_1(U) \\ \psi([a]_V) = j_2[a]_V & [a]_V \in \pi_1(V) \end{cases}$$

(i) ψ is epi (onto)

($\pi_1(X)$ is gen'd by $\pi_1(U), \pi_1(V)$ - seen before)

(ii) $N = \text{norm} \langle [i_1(b)] [i_2(b)]^{-1}; b \in \pi_1(U \cup V) \rangle$

$$N \subset \ker(\psi).$$

$$i: U \cup V \rightarrow X$$

Proof If $b \in \pi_1(U \cup V)$, $j_1 \circ i_1[b] = i[b] = j_2 \circ i_2[b]$

$$\begin{aligned} \psi[i_1(b)] &= j_1[i_1(b)] = (j_1 \circ i_1)[b] \\ \psi[i_2(b)] &= j_2[i_2(b)] = (j_2 \circ i_2)[b] \end{aligned} \quad \text{) equal}$$

so $\psi[i_1(b)] [i_2(b)]^{-1} = e_{x_0}$ $e_{x_0} \in \pi_1(X)$

(proves $N \subset \ker \psi$)

Claim: in fact $\ker \psi = N$ (this implies $\tilde{\psi}: H \rightarrow \pi_1(X)$ is iso)

$$\tilde{\psi}: \pi_1(U) * \pi_1(V) / N \rightarrow \pi_1(X) \text{ well-def (since } \ker \psi \supset N)$$

$$\tilde{\psi}(gN) \stackrel{\text{def}}{=} \psi(g)$$

ETS (11) $\Phi \circ \tilde{\psi} = \text{id}$ in $H = \pi_1(U) * \pi_1(V) / N$.

Then if $g \in \ker \psi$, $\tilde{\psi}(gN) = e_{x_0}$ so $\Phi(\tilde{\psi}(gN)) = e_H$

If $g \in \pi_1(U)$ $\tilde{\psi}(gN) = \psi(g) = j_1(g) \in \pi_1(X) \Rightarrow g \in N$

$$(\Phi \circ \tilde{\psi})(gN) = \Phi(j_1(g)) = \phi_1(g) = gN$$

If $g \in \pi_1(V)$: $(\Phi \circ \tilde{\psi})(gN) = gN$ (proves (ii)).

□ (S-VK) thm.

Cor./Appl's

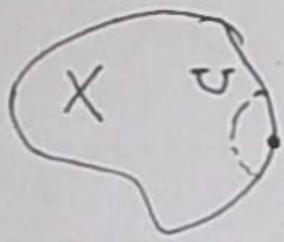
① $\pi_1(U \cup V) = \{e_{x_0}\} \rightarrow \pi_1(X) = \pi_1(U) * \pi_1(V)$



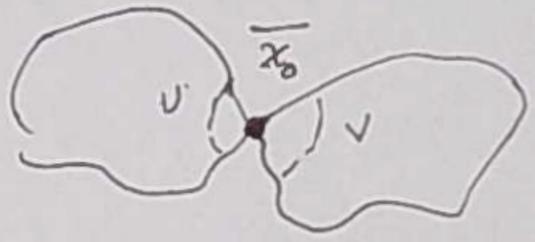
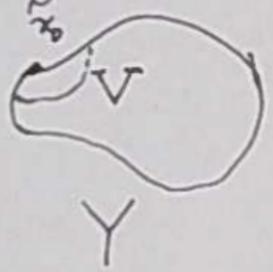
$\pi_1(S^1 \vee S^1) = F_2$ (free gp on 2 generators)

② X, Y spaces w/ basepoint (ex: X, Y manifolds)

Ass. X, Y have basepoint contractible nbhd $\text{\textcircled{P}}$



$\exists U \subset X$ nbd of x_0 , contractible to $\{x_0\}$



$X \vee Y$

Then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$

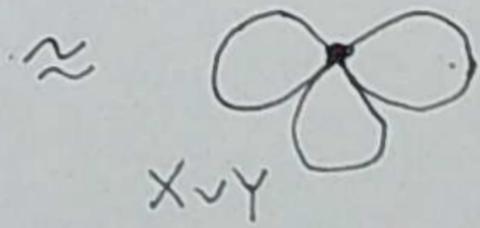
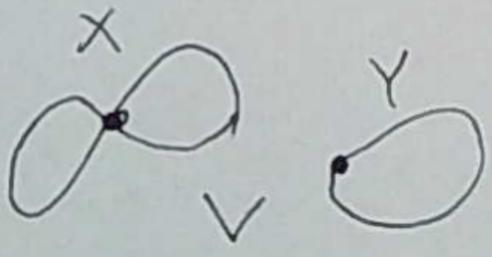
Apply S-vK thm to $U_X = X \cup V$ (abeng. nbhd)

and $V_Y = Y \cup U$

then $U_X \cup V_Y = X \vee Y$ (both open)

$U_X \cap V_Y$ is contractible to $\bar{x}_0 \rightarrow \pi_1(U_X \cap V_Y) = \{e_{\bar{x}_0}\}$

③ X is a def. retract of U_X $\pi_1(X) \approx \pi_1(U_X)$
 Y is a def. retract of V_Y $\pi_1(Y) \approx \pi_1(V_Y)$



π_1 (bouquet of n loops)
 \cong
 F_n (free gp on n gens)