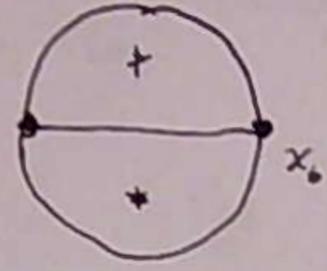
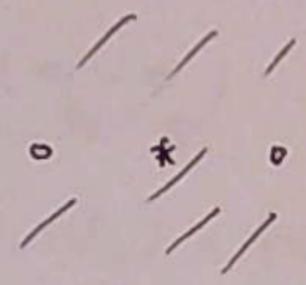




$S^1 \vee S^1$



$\Theta$  "theta"



$\mathbb{R}^2 \setminus \{p, q\}$

Rk. Deformation retractions  $r: X \rightarrow A$  are homotopy equivalences.  
 (  $r \circ i_A = id_A$ ,  $i_A \circ r \sim id_X$  )  $i_A: A \rightarrow X$  inclusion

Both  $S^1 \vee S^1$  &  $\Theta$  are def. retracts of  $\mathbb{R}^2 \setminus \{p, q\}$   
 so  $S^1 \vee S^1$  &  $\Theta$  are homotopy eq. (Not homeomorphic!)

Serpent-v-Kampfen thru

$X = U_1 \cup U_2$   $U_i = X$  open  $x_0 \in U_1 \cap U_2$   
 $U_1 \cap U_2, U_i, X$  path-con.

Q) Relate  $\pi_1(X)$  to  $\pi_1(U_1), \pi_1(U_2)$  [omit basept  $x_0$ ]  
 $\pi_1(U_1), \pi_1(U_2)$  generate  $\pi_1(X)$   
free product of 2 gps  $G_1, G_2$ .

$G_1 * G_2 = \{ g_1 g_2 \dots g_N \cup \{e\} \}$   $g_i$  in  $G_1$  or  $G_2$ ;  $g_i, g_{i+1}$  not in same gp,  $g_i \neq e, e_2$

gp structure  
 $(g_1 h_1 g_2) (g_2^{-1} h_2^{-1} g_3) = g_1 \underbrace{(h_1 h_2^{-1})}_h g_3 = g_1 h g_3$   
 (concatenate/reduce)

We have a hom.  $G_1 = \pi_1(U_1), \pi_1(U_2) \cong G_2$

$\xrightarrow{\text{extend}} \pi_1(U_1) * \pi_1(U_2) \xrightarrow{\text{inclusion}} \pi_1(X)$  ONTO  
 the inclusion hom's  $f_1: \pi_1(U_1) \rightarrow \pi_1(X)$   
 $f_2: \pi_1(U_2) \rightarrow \pi_1(X)$

not

$$\begin{array}{c}
 G_1 \xrightarrow{k_1} G_1 * G_2 \\
 G_2 \xrightarrow{k_2} G_1 * G_2
 \end{array}
 \text{ (subgroups)}$$

$k_1, k_2$ : inclusion hom.  
 $(e_1 \sim e, e_2 \sim e)$

are these groups ISO? (Yes if  $\pi_1(U_1 \cap U_2) = \{e\}$ )  
 $\pi_1(U_1) + \pi_1(U_2)$  and  $\pi_1(X)$

Geometric part

Look for an "inverse hom."

$$\pi_1(X) \longrightarrow \pi_1(U_1) * \pi_1(U_2)$$

start w/ loops in  $U_1$  or  $U_2$

$$\pi_1(U_1) \longrightarrow \pi_1(U_1) * \pi_1(U_2) \text{ (inclusion)}$$

$$\pi_1(U_2) \longrightarrow \pi_1(U_1) * \pi_1(U_2)$$

well defined? what if the loop  $f \in \Omega_{x_0}(X)$  takes values in  $U_1 \cap U_2$ ?

Let  $i_1 : \pi_1(U_1 \cap U_2) \longrightarrow \pi_1(U_1)$   
 $i_2 : \pi_1(U_1 \cap U_2) \longrightarrow \pi_1(U_2)$  (inclusion hom.)

Consider

$$N \subset \pi_1(U_1) * \pi_1(U_2)$$

$N$  = normal subgroup generated by all elements of the form

$$\left\{ (k_1 \circ i_1)(b) \left[ (k_2 \circ i_2)(b) \right]^{-1}; b \in \pi_1(U_1 \cap U_2) \right\}$$

Consider the group

$$H = \pi_1(U_1) * \pi_1(U_2) / N$$

We obtain hom

$$\phi_1 : \pi_1(U_1) \longrightarrow H, \phi_2 : \pi_1(U_2) \longrightarrow H \text{ (inclusion followed by proj.)}$$



goal build  $\Phi : \pi_1(X) \longrightarrow H$  hom.

$$\text{s.t. } \Phi \circ j_i = \phi_i, \Phi \circ j_2 = \phi_2.$$

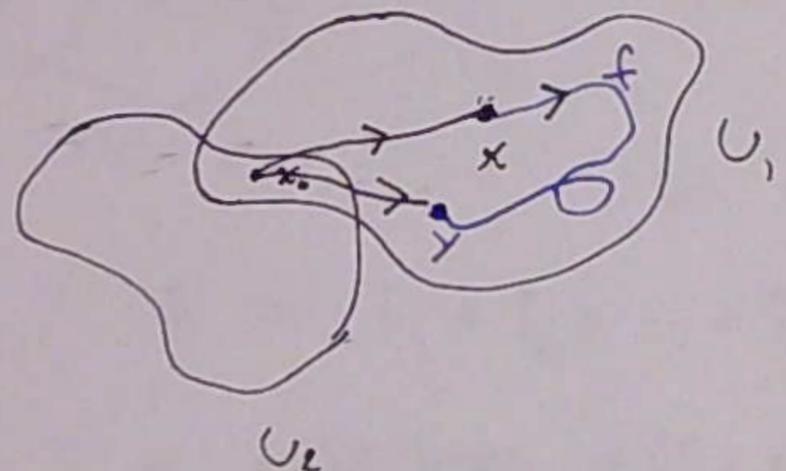
**Step 1** If  $f$  is a loop (at  $x_0$ ), taking values in  $U_1$  or  $U_2$ ,

let  $\rho([f]) = \phi_1([f])$  (or  $\phi_2([f])$ )  $\in H$

- $\rho$  dep. only of  $[f]$  ( $\phi_1, \phi_2$  ~~are~~ <sup>since</sup> def. only on  $[f]_{U_1}, [f]_{U_2}$ )
- $\rho$  preserves concatenation ( $\phi_1, \phi_2$  <sup>since</sup> are hom.)
- $\rho$  is well-def: if  $f$  is a loop in  $U_1 \cap U_2$

$b = [f]_{U_1 \cap U_2}$   $k_1 \circ i_1(b) = k_2 \circ i_2(b)$  since  $f \in \Omega_x(U_1 \cap U_2)$   
 $\phi_1(b) = \phi_2(b)$  since  $k_1(b), k_2(b)$  project to same element of  $H$ .

**Step 2** Extend  $\rho$  to a map  $\sigma$  to  $H$ , defined on paths taking values in  $U_1$  or  $U_2$ .



pick paths  $\alpha_x$  (for each  $x \in X$ ) from  $x_0$  to  $x$   
 $\alpha_x$  takes values in  $U_1$  (resp.  $U_2, U_1 \cap U_2, \{x_0\}$ )  
 if  $x \in U_1$  (resp.  $U_2, U_1 \cap U_2, \{x_0\}$ )

given  $f$ : path in  $U_1$   $f: x \rightsquigarrow y$

consider  $L(f) = (\alpha_x * f) * \bar{\alpha}_y \in \Omega_{x_0}(U_1)$   
 (similarly if  $f$  is a path in  $U_2$ )

Let  $\sigma: P(U_1) \cup P(U_2) \rightarrow H$   
 $\sigma(f) = \rho(L(f))$

(1)  $\sigma$  dep. only on the path homotopy class of  $f$   
 $f \simeq g$  (path)  $\Rightarrow L(f) \simeq L(g)$  (in  $\Omega_{x_0}(U_i)$ )  $[f] = [L(g)]$   
 $\Rightarrow \sigma(f) = \sigma(g)$  (since  $\rho$  dep. only on homotopy class)

(2)  $\sigma(f * g) = \sigma(f) \cdot \sigma(g)$  (product in  $H$ )  
 $f, g \in P(U_1)$  or  $P(U_2)$

Proof of (2)

$$f: X \rightarrow Y \quad g: Y \rightarrow Z \quad (f * g \text{ well-def.})$$

(4)

$$L(f) * L(g) = ((\alpha_x * f) * \bar{\alpha}_y) * ((\alpha_y * g) * \bar{\alpha}_z)$$

$$\simeq \alpha_x * (f * g) * \bar{\alpha}_z$$

$$= L(f * g)$$

(assoc of concatenation at level of htopy)

$p$  compatible w/ concatenation

$$\sigma(f * g) = p(L(f * g)) = p(L(f) * L(g)) = p(L(f)) * p(L(g))$$

$$= \sigma(f) * \sigma(g)$$

( $p$  dep. only on htopy class)

$\hookrightarrow$  product in  $H$

Step 3

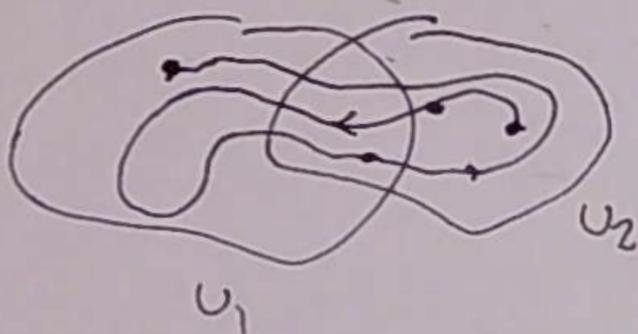
Extend  $\sigma$  to a map  $\tau$  defined on paths in  $X$

$$f \in \mathcal{P}(X)$$

subdivision of  $[0, 1]$

$$\xi = 0 < s_1 < \dots < s_n = 1$$

s.t.  $f|_{J_i}$  takes values in  $U_1$  or  $U_2$



Let  $f_i: [0, 1] \rightarrow X$  be the reparam of  $f|_{J_i}$

$$f \simeq f_1 * f_2 * \dots * f_n$$

$$\text{Let } \tau = \sigma(f_1) * \sigma(f_2) * \dots * \sigma(f_n) \in H$$

well-defined? (adding a pt to the partition  $(s_i)$  doesn't change the def.)

two things to check

(1)  $\tau$  dep. only on  $[f]$  (path homotopy class)

(2)  $\tau(f * g) = \tau(f) * \tau(g)$  (product in  $H$ )

to be continued...